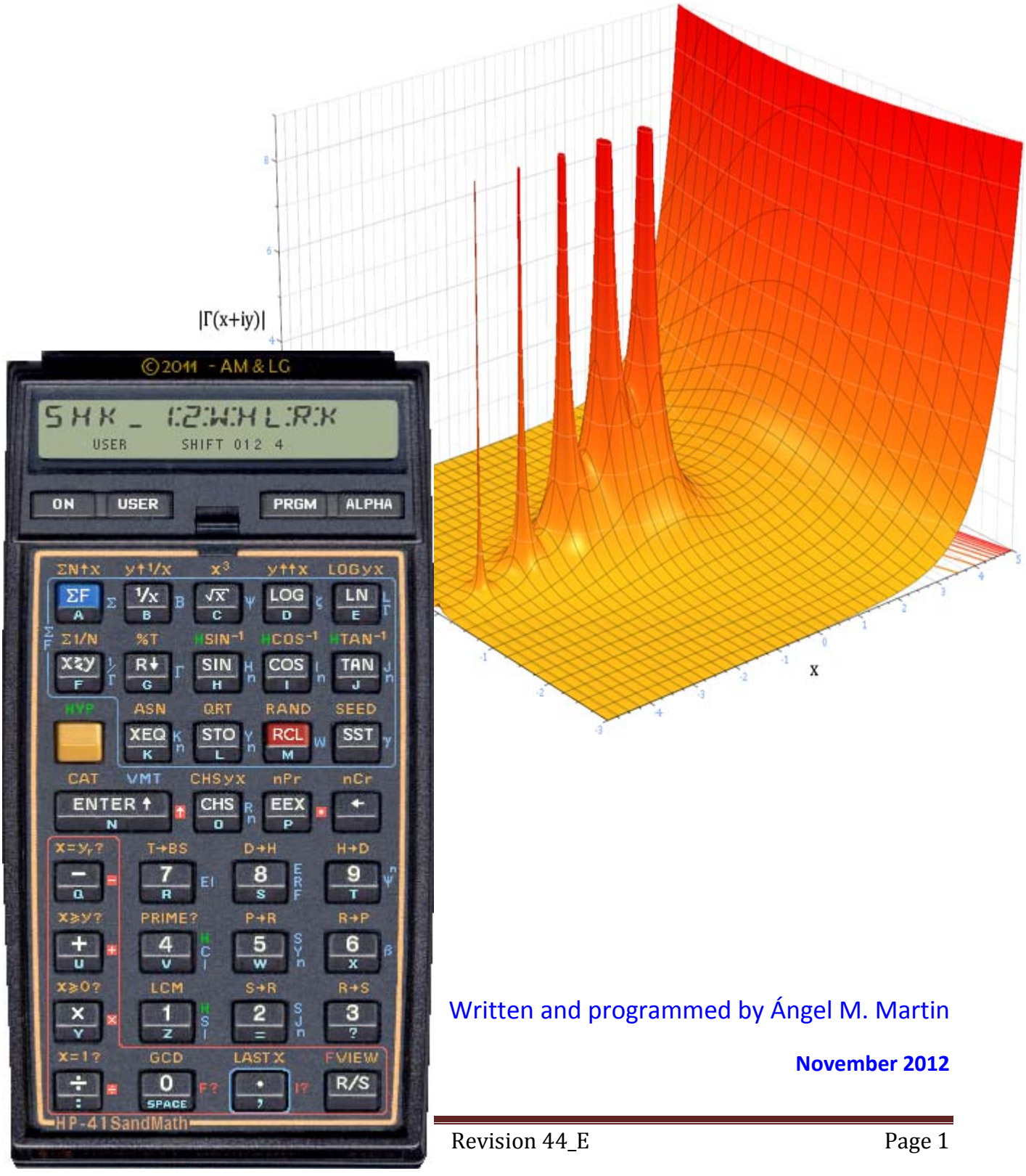
SANDMATH_44

Math Extensions for the HP-41

User's Manual and Quick Reference Guide



Written and programmed by Ángel M. Martin

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This compilation revision 4.44.44 (really!)

Acknowledgments.-

Documentation wise, this manual begs, steals and borrows from many other sources – in particular Jean-Marc Baillard's program collection on the web. Really both the SandMath and this manual would be a much lesser product without Jean-Marc's contributions.

There are multiple graphics and figures taken from Wikipedia and Wolfram Alpha, notably when it comes to the Special Functions sections. I'm not aware of any copyright infringement, but should that be the case I'll of course remove them and create new ones using the SandMath function definition and **PRPLOT**. Just kidding...

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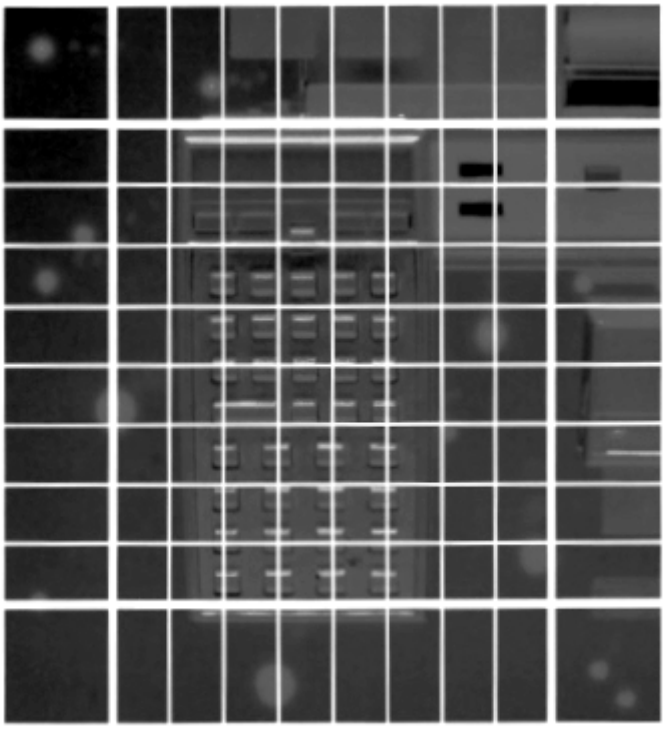


Note: Make sure the revision "G" (or higher) of the Library#4 module is installed.

SandMath_44 Module – Revision E
Math Extensions for the HP-41 System

1. Introduction.

Simply put: here's a compilation of (mostly MCODE) Math functions to extend the native function set of the HP-41 system. At this point in time - way over 30 years after the machine's launch - it's more than likely not realistic to expect them to be profusely employed in FOCAL programs anymore - yet they've been included for either intrinsic interest (read: challenging MCODE or difficult to realize) or because of their inherent value for those math-oriented folks.

This module is an 8k implementation. The first 4k includes more general-purpose functions, re-visiting the usual themes: Fractions, Base conversion, Hyperbolic functions, RCL Math extensions, as well as simple-but-neat little gems to round off the page. In sum: all the usual suspects for a nice ride time.

The second page delves into deeper territory, touching upon the special functions field and Probability/Statistics. Some functions are plain "catch-up" for the 41 system (sorely lacking in its native incarnation), whilst others are a divertment into a tad more complex math realms. All in all a mixed-and-matched collection that hopefully adds some value to the legacy of this superb machine – for many of us the best one ever.

I am especially thankful for the essential contributions from Jean-Marc Baillard: more than 3/4ths of this module are directly attributable to his original programs, one way or another.

Wherever possible the 13-digit OS routines have been used throughout the module – ensuring the optimal use of the available resources to the MCODE programmer. This prevents accuracy loss in intermediate calculations, and thus more exact results. For a limited precision CPU (certainly per today's standards) the Coconut chip still delivers a superb performance when treated nicely.

The module uses routines from the Page#4 Library (a.k.a. "Library#4"). Many routines in the library are general-purpose system extensions, but some of them are strictly math related, as auxiliary code repository to make it all fit in an 8k footprint factor - and to allow reuse with other modules. This is totally transparent to the end user, just make sure it is installed in your system and that the revisions match. See the relevant Library#4 documentation if interested.

Function Launchers and Mass key assignments.

As any good "theme" module worth its name, the SandMath has its own mass-Key assignment routine. Use it to assign the most common functions within the ROM to their dedicated keys for a convenient mapping to explore the functions. Besides that, a distinct feature of this module is the function launchers, used to access diverse functions grouped by categories. These include the Hyperbolic, the Fractions, the RCL Math, and the Special Function groups. This saves memory registers for key assignments, whilst maintaining the standard keyboard available also in USER mode for other purposes.

This is the fourth incarnation of the SandMath project, which in turn has had a fair number of revisions and iterations on its own. *The new distinct addition has been a secondary Function address Table (FAT)* to provide access to more functions, exceeding the limit per page imposed by the operating system. Some other refinements consisted in a rationalization of the backbone architecture, as well as a more modular approach to each of pages of the module. Gone are the "8k" vs. "12k" distinctions of the past – as now the Matrix and Polynomial functions have an independent life of their own in separate modules - more on that to come.

This manual is a more-or-less concise document that only covers the normal use of the functions.

Conventions used in this manual.

All throughout this manual the following convention will be used in the function tables to denote the availability of each function in the different function launchers:

[*]:	assigned to the keyboard by	**HMKEYS**
[ΣF]:	direct execution from the main launcher	**ΣFL**
[**H**]:	executed from the hyperbolics launcher	**-HYP**
[**F**]:	executed from the fractions launcher	**-FRC**
[RC]:	executed from the RCL# launcher,	**-RCL**
[CR];	executed from the Carlson Launcher	(no separate function exists)
[HK]:	executed from the Hankel launcher	(no separate function exists)
[ΣΣ]:	executed from the Statistics Menu,	**–ST/PRB**
[Σ$]:	sub-function in the secondary FAT.	**ΣFL$**

HMKEYS uses the value in X as a flag to decide whether to assign or to remove the mass key assignments. If x=0 the assignments will be removed, any other value will make them. There are a total of 25 functions assigned, refer to the SandMath overlay for details.

Note for Advanced Users:

Even if the SandMath is an 8k module, it is possible to configure only the first (lower) page as an independent ROM. This may be helpful when you need the upper port to become available for other modules (like mapping the CL's MMU to another module temporarily); or permanently if you don't care about the High Level Math (Special Functions) and Statistics sections.

Think however that the FAT entries for the Function launchers are in the upper page, so they'll be gone as well if you use the reduced foot-print (4k) version of the SandMath.

Upper Page:	High-Level Math, Stats, Function Launchers
Lower Page:	SandMath_44, FRC, HYP, RCL# Math

Note that it is not possible to do it the other way around; that is plugging only the upper page of the module will be dysfunctional for the most part and likely to freeze the calculator– do not attempt.

Final Disclaimer.-

With "just" an EE background the author has had his dose of relatively special functions, from college to today. However not being a mathematician doesn't qualify him as a field expert by any stretch of the imagination. Therefore the descriptions that follow are mainly related to the implementation details, and not to the general character of the functions. This is not a mathematical treatise but just a summary of the important aspects of the project, highlighting their applicability to the HP-41 platform.

Getting Started: Accessing the Functions.

There are about 160 functions in the SandMath Module. With each of its two pages containing its own function table, this would only allow to index 128 functions - where are the others and how can they be accessed? The answer is called the "Multi-Function" groups.

Multi-Functions **ΣFL#** and **ΣFL$** provide access to an entire group of sub-functions, grouped by their affinity or similar nature. The sub-functions can be invoked using either its index within the group, using **ΣFL#**, or its direct name, using **ΣFL$.** This is implemented in such a way that they are also programmable, and can be entered into a program line using a technique called "*non-merged functions*".

You may already be familiar with this technique, originally developed by the HEPAX programmers. In the HEPAX there were two of those groups; one for the XF/M functions and another for the HEPAX/A extensions. The PowerCL Module also contains its own, and now the SandMath joins them – this time applied to the mathematical extensions, particularly for the Special Functions group.

A sub-function catalog is also available, listing the functions included within the group. Direct execution (or programming if in PRGM mode) is possible just by stopping the catalog at a certain entry and pressing the **XEQ** key. The catalog behaves very much live the native ones in the machine: you can stop them using R/S, SST/BST them, press ENTER^ to move to the next "sub-section", cancel or resume the listing at any time.

As additional bonus, the sub-function launcher **ΣFL$** will also search the "main" module FAT if the sub-function name is not found within the multi-function group – so the user needn't remember where a specific function sought for was located. In fact, **ΣFL$** will also "find" a function from any plugged-in module in the system, even outside of the SandMath module.

Main Launcher and Dedicated Launchers.

The Module's Main launcher is [**ΣFL**]. Think of it as the trunk from which all the other launchers stem, providing the branches for the different functions in more or less direct number of keystrokes. With a well-thought out logic in the functions arrangement then it's much easier to remember a particular function placement, even if its exact name or spelling isn't know, without having to type it or being assigned to any key.

Despite its unassuming character, the ΣFL prompt provides direct access to many functions. Just press the appropriate key to launch them, using the SandMath Overly as visual guide: the individual functions are printed in BLUE, with their names set aside of the corresponding key. They become active when the "**ΣF _**" prompt is in the display.

, or


Besides providing direct access to the most common Special Functions, **ΣFL** will also trigger the dedicated function launchers for other groups, like **-HYP**, **-FRC**, **CR**, **HK**, **-STAT**, and **ΣFL$** itself. Think of these groupings as secondary "menus" and you'll have a good idea of their intended use. The following keys activate the secondary menus:

[**A**], activates the STAT/PRB menus.
[**H**], activates the HK _ launcher
[**O**], activates the CR _ launcher
[**.**] , radix activates the FRC launcher
[SHIFT] switches into the hyperbolic choices, Use [SHIFT] to toggle direct/inverse
[ALPHA] activates the **ΣFL$** sub-functions launcher
[**<-**], back-arrow cancels it or returns to it from a secondary menu.

Like the native implementation, the name of the launched function will be shown in the display while you hold the corresponding key – and NULLED if kept pressed. This provides visual feedback on the action for additional assurance.

Typically the secondary launchers have the possible choices in their prompt, we'll see them later on. The STAT menu differs from the others in that it consists of two line-ups toggled with the [SHIFT] key – providing access to 10 functions using the keys in the top-row directly below the function symbol.

This is a good moment to familiarize yourself with the [ΣF] launcher. Go ahead and try it, using it also in PRGM mode to enter the functions as program lines. Note that when activating **ΣFL$** you'll need to press [ALPHA] a second time to spell the sub-function name.

Direct-access function keys, in alphabetical order:

[**A**]: Stat/Prob MENUS
[B]: Euler's Beta
[C]: Digamma (PSI)
[D]: Rieman's Zeta
[E]: Gamma Natural log
[F]: Inverse Gamma
[G]: Euler's Gamma
[**H**]: Hankel's Launcher
[I]: Bessel I(n,x)
[J]: Bessel J(n,x)
[SHIFT]: Hyperbolics Launcher
[K]: Bessel K(n,x)
[L]: Bessel Y(n,x)
[M]: Lambert's W
[SST]: Incomplete Gamma
[N]: FCAT, sub-functions CATalog
[**O**]: Carlson Launcher
[R]: Exponential integral
[S]: Error Function
[T]: Polygamma (PsiN)
[V]: Cosine Integral
[W]: Spherical Y(n,x)
[X]: Incomplete Beta
[Z]: Sine Integral
[=]: Spherical J(n,x)
[,]: Fractions Launcher
[R/S]: View Mantissa
[**<-**]: Cancels out to the OS
[ALPHA]: Sub-function Launcher
[ON]: Turns the calculator OFF

A green "**H"** on the overlay prefixing the function name represents the Hyperbolic functions. This also includes the Hyperbolic Sine and Cosine integrals, in addition to the three "standard" ones. Using the [SHIFT] key will toggle between the direct and inverse functions. Pressing [**<-**] will take you back to the main **ΣFL** prompt.

The Fraction functions are encircled by a red line on the overlay, at the bottom and left rows of the keyboard. They include the fraction math, plus a fraction Viewer and fraction/Integer tests.

The Hankel and Carlson launchers will present their choices in their prompts, and will be covered later in the manual.

Note that the RCL Math functions are also linked to the main launcher, to invoke them use the [RCL] launcher, sort of "Hyper-RCL" thus need to press: [ΣFL], [**HYP**] to get the "-RCL# _ _" prompt.

Appendix 1.- Launcher Maps.

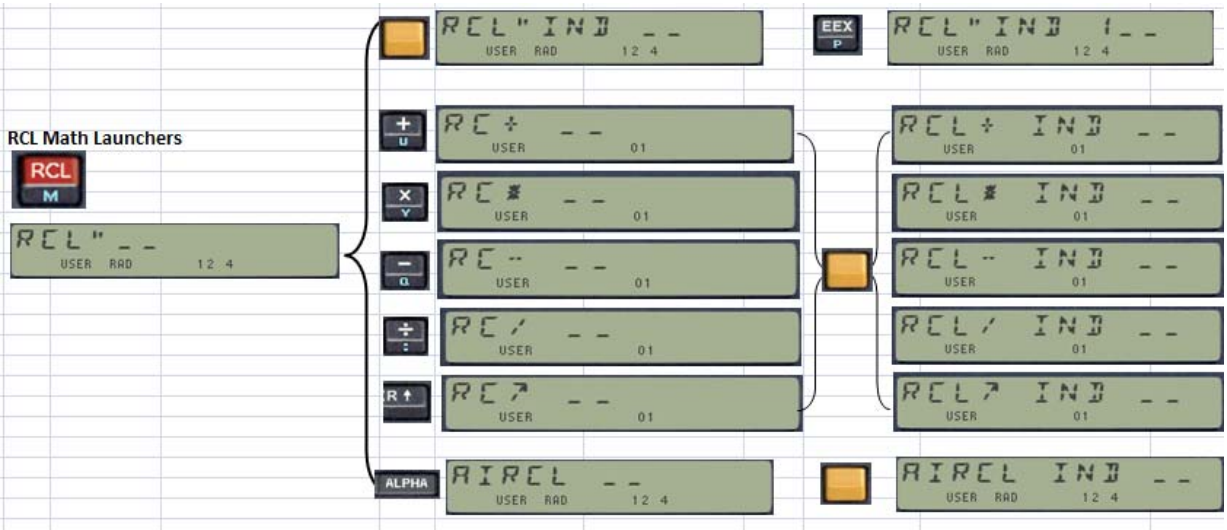
The figures below provide a better overview, illustrating the hierarchy between launchers and their interconnectivity. For the most part it is always possible to return to the main launcher pressing the back arrow key, improving so the navigation features – rather useful when you're not certain of a particular function's location.

The first one is the Main SandMath Launcher.

The first mapping doesn't show all the direct execute function keys. Use the SandMath overlay as a reference for them (names written in BLUE aside the functions).

Note that **ΣFL$** will require pressing ALPHA a second time in order to type the sub-function name.

And here's the Enhanced RCL MATH group:



Here all the prompts expect a numeric entry. The two top rows keys can be used as shortcuts for 1-10. Note that No **STK** functionality is implemented – even if you can force the prompt at the **IND** step. Typically you'll get a *DATA ERROR* message - Rather not try it :-)

Function index at a glance.

And without further ado, here's the list of functions included in the module. First the main functions:

#	Name	Description
0	-SNDMATH-44	*Displays "RUNNING..."*
1	2^X-1	Powers of 2
2	Σ1/N	Harmonic Numbers
3	ΣDGT	Sum of mantissa digits
4	ΣN^X	Geometric Sums
5	AINT	Alpha Integer Part
6	ATAN2	Dual-argument ATAN
7	BS>D	Base to Dec
8	CBRT	Cubic Root
9	CEIL	Ceil function
10	CHSYX	CHSY by X
11	CROOT	Cubic Equation Roots
12	CVIETA	Driver for CROOT
13	D>BS	Dec to Base
14	D>H	Dec to Hex
15	E3/E+	1,00X
16	FLOOR	Floor Function
17	GEU	Euler's Constant
18	H>D	Hex to Dec
19	HMS*	HMS Multiply
20	HMS/	HMS Divide
21	LOGYX	LOG b of X
22	MANTXP	Mantissa
23	MKEYS	Mass Key Assgn.
24	P>R	Complete P-R
25	QREM	Quotient Reminder
26	QROOT	2nd. Degree Roots
27	QROUT	Outputs Roots
28	R>P	Complete R-P
29	R>S	Rectangular to Spherical
30	S>R	Spherical to Rectangular
31	STLINE	Straight Line from Stack
32	T>BS _ _	Dec to Base
33	VMANT	View Mantissa
34	X^3	X^3
35	X=1?	Is X=1?
36	X=Y?R	Is X~Y? (rounded)
37	X>=0?	is X>=0?
38	X>=Y?	is X>=Y?
39	Y^1/X	Xth. Root of Y
40	Y^^X	Extended Y^X
41	YX^	Modified Y^X
42	-FRC _	*Fraction Math Launcher*
43	D>F	Decimal to Frac
44	F+	Fraction Addition
45	F-	Fraction Subtract
46	F*	Fraction Multiply
47	F/	Fraction Divide
48	FRC?	is X fractional?
49	INT?	Is X Integer?
50	-HYP _	*Hyberbolics Launcher*
51	HACOS	Hypebolic ACOS

#	Name	Description
0	-HL MATH _	*Secton Header*
1	1/GMF	Reciprocal Gamma (Cont..Frc.)
2	ΣFL	Main Function Launcher
3	ΣFL$	Launcher by Name
4	ΣFL#	Launcher by index
5	BETA	Beta Function
6	CI	Cosine Integral
7	EI	Exponential Integral
8	ELIPF	Eliptic Integral 1st. Kind
9	ERF	Error Function
10	FFOUR	Fourier Series
11	GAMMA	Gamma Function (Lanczos)
12	HCI	Hyperbolic Cosine Integral
13	HGF+	Generalized Hypergeometric Function
14	HSI	Hyperbolic Sine Integral
15	IBS	Bessel In Function
16	ICBT	Incomplete Beta Function
17	ICGM	(Lower) Incomplete Gamma Function
18	JBS	Bessel Jn Function
19	KBS	Bessel Kn Function
20	LINX	Polylogarithm
21	LNGM	Logarithm Gamma Function
22	PSI	Digamma Function
23	PSIN	Polygamma
24	PP2	Point-to-Point Dist
25	POCH	Pochhammer Symbol
26	SI	Sine Integral
27	SIBS	Spherical I Bessel
28	SJBS	Spherical J Bessel
29	SYBS	Spherical Y Bessel
30	WL0	Lambert W Function
31	WL1	Lambert W Function
32	YBS	Bessel Yn
33	ZETA	Zeta Function (Direct method)
34	ZETAX	Zeta Function (Borwein)
35	ZOUT	Output Complex to ALPHA
36	DECX	Decrease X
37	DECY	Decrease Y
38	INCX	Increase X
39	INCY	Increase Y
40	-PRB/STS _	*Displays STAT menu*
41	%T	Percentual
42	CORR	Correlation Coefficient
43	COV	Sample Covariance
44	DSP?	Display Digits
45	EVEN?	is X Even?
46	GCD	Greatest Common Divisor
47	LCM	Least Common Multiple
48	LGMF	Log Multi-Factorial
49	LR	Linear Regression
50	LRY	LR Y-value
51	NCR	Permutations

52	HASIN	Hyperbolic ASIN	52	NPR	Combinations
53	HATAN	Hyperbolic ATAN	53	ODD?	Is X Odd?
54	HCOS	Hyperbolic COS	54	PDF	Probability Distribution Function
55	HSIN	Hyperbolic SIN	55	PFCT	Prime Factorization in Alpha
56	HTAN	Hyperbolic TAN	56	PRIME?	Is X Prime?
57	-RCL _	*Extended Recall*	57	RAND	Random Number
58	AIRCL	ARCL Integer Part	58	RGMAX	Block Maximum
59	RCL^	Recall Power	59	RGSORT	Register Sort
60	RCL+	Recall Add	60	SEEDT	Stores Seed for RNDM
61	RCL-	Recall Subtract	61	ST<>Σ	Exchange ST & ΣREG
62	RCL*	Recall Multiply	62	STSORT	Stack Sort
63	RCL/	Recall Divide	63	XFACT	Extended FACTorial

Functions in blue are all in MCODE.
Functions in black are MCODE entries to FOCAL programs.

And now the sub-functions within the Special Functions Group – deeply indebted to Jean-Marc's contribution (and not the only section in the module). Note there are two sections within this auxiliary FAT – you can use the **FCAT** hot keys to navigate the groups.

Name	Description	*Author*
-SP FNC	Cat header - does FCAT	*Ángel Martin*
#BS	Aux routine, All Bessel	*Ángel Martin*
#BS2	Aux routine 2nd. Order, Integers	*Ángel Martin*
AIRY	Airy Functions Ai(x) & Bi(x)	*JM Baillard*
ALF	Associated Legendre function 1st kind - Pnm(x)	*JM Baillard*
AWL	Inverse Lambert	*Ángel Martin*
CRF	Carlson Integral 1st. Kind	*JM Baillard*
CRG	Carlson Integral 2nd. Kind	*JM Baillard*
CRJ	Carlson Integral 3rd. Kind	*JM Baillard*
CSX	Fresnel Integrals, C(x) & S(x)	*JM Baillard*
DAW	Dawson integral	*JM Baillard*
DBY	Debye functions	*JM Baillard*
ELIPF	Eliptic Integral	*Ángel Martin*
HGF	Hypergeometric function	*JM Baillard*
HK1	Hankel1 Function	*Ángel Martin*
HK2	Hankel2 Function	*Ángel Martin*
HNX	Struve H Function	*JM Baillard*
ITI	Integral if IBS	*Ángel Martin*
ITJ	Integral of JBS	*Ángel Martin*
KLV	Kelvin Functions 1st kind	*JM Baillard*
KUMR	Kummer Function	*Ángel Martin*
LERCH	Lerch Transcendent function	*JM Baillard*
LI	Logarythmic Integral	*Ángel Martin*
LNX	Struve Ln Function	*JM Baillard*
LOML	Lommel s1 function	*JM Baillard*
RHGF	Regularized hypergeometric function	*JM Baillard*
SAE	Surface Area of an Ellipsoid	*JM Baillard*
SHK1	Spherical Hankel1	*Ángel Martin*
SHK2	Spherical Hankel2	*Ángel Martin*
TMNR	Toronto function	*JM Baillard*
WEBAN	Weber and Anger functions	*JM Baillard*
W0L	Lambert W0	*Ángel Martin*
W1L	Lambert W1	*Ángel Martin*

The last section groups the factorial functions, circling back from the special functions into the number theory field - a timid foray to say the most.

Name	Description	*Author*
-FACTORIAL	Section Header	n/a
FFCT	Falling Factorial	*Ángel Martin*
POCH	Pochhammer symbol	*Ángel Martin*
MFCT	Multi-Factorial	*JM Baillard*
LGMF	Logarithm Multi-Factorial	*JM Baillard*
PSD	Poisson Standard Distribution	*Ángel Martin*
SFCT	Super Factorial	*JM Baillard*
XFCT	Extended Factorial	*Ángel Martin*
FCAT (*)	Function Catalogue	*Ángel Martin*

(*) The best way to access **FCAT** is through the main launcher [**ΣFL**] , then pressing ENTER^ ("**N**")

FCAT (and **–SP FNC**) are usability enhancements for the admin and housekeeping. It invokes the sub-function CATALOG, with *hot-keys for individual function launch and general navigation*. Users of the POWERCL Module will already be familiar with its features, as it's exactly the same code – which in fact resides in the Library#4 and it's reused by both modules (so far).

Its hot-keys and actions are listed below:

[**R/S**]: halts the enumeration
[**SST/BST**]: moves the listing one function up/down
[SHIFT]: sets the direction of the listing forwards/backwards
[**XEQ**]: direct execution of the listed function – or entered in a program line
[**ENTER^**]: moves to the next/previous section depending on SHIFT status
[**<-**]: back-arrow cancels the catalog

One limitation of the sub-functions scheme that you'll soon realize is that, contrary to the standard functions, *they cannot be assigned to a key for the USER keyboard*. Typing the full name (or entering its index at the ΣFL# prompt) is always required. This can become annoying is you want to repeatedly execute a given sub- function.

A work-around this consists of writing a micro-FOCAL program with just the sub-function as a single pair of program lines, and then assign it to the key of choice. Not perfect but it works.

Note: Make sure the revision "G" (or higher) of the Library#4 module is installed.

2. Lower-Page Functions in detail

The following sections of this document describe the usage and utilization of the functions included in the SandMath_44 Module. While some are very intuitive to use, others require a little elaboration as to their input parameters or control options, which should be covered here. Reference to the original author or publication is always given, for additional information that can (and should) also be consulted.

2.1. SANDMATH GROUP	

The Module starts with an assorted group of functions providing simple but important additions to the native function set.

2.1.1. Elementary Math functions

Even the most complex project has its basis – simple enough but reliable, so that it can be used as solid foundation for the more complex parts. The following functions extend the HP-41 Math function set, and many of them will be used either as MCODE subroutines or directly in FOCAL programs.

	Function	Author	Description
	2^X-1	*Ángel Martin*	Self-descriptive, faster and better precision than FOCAL
[*]	**Σ1/N**	*Ángel Martin*	Harmonic Number H(n)
	ATAN2	*Ángel Martin*	Two-argument arctangent
[*]	**CBRT**	*Ángel Martin*	Cubic root (main branch)
	CEIL	*Ángel Martin*	Ceiling function of a number
[*]	**CHSYX**	*Ángel Martin*	Multiple CHS by Y
	E3/E+	*Ángel Martin*	Index builder
	FLOOR	*Ángel Martin*	Floor function of a number
	GEU	*Ángel Martin*	Euler-Mascheroni constant
[*]	**LOGYX**	*Ángel Martin*	Base-Y Natural logarithm of X
	QREM	*Ken Emery*	Quotient Remainder
[*]	**X^3**	*Ángel Martin*	Cube power of X
[*]	**Y^1/X**	*Ángel Martin*	x-th root of Y
[*]	**Y^^X**	*Ángel Martin*	Very large powers of X (result >= 1E100)
	YX^	*JM Baillard*	Modified Y^X (does 0^0=1)

2^X-1 provides a more accurate result for smaller arguments than the FOCAL equivalents. It will be used in the ZETAX program to calculate the Zeta function using the Borwein algorithm.

Σ1/N calculates the Harmonic number of the argument in X, as is the sum of the reciprocals of the natural numbers lower and equal than n:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

It will be used to calculate the Bessel functions of the second kind, K(n,x) and Y(n,x).

Also related to the same problem, and in general relevant to the summation of alternating series, is the function **CHSYX -** an extension of **CHS** but dependent of the number in Y. Its expression is:

CHS(Y,X)= x*(-1)^y, returning +/- X, depending on whether the number in Y is even or odd respectively.

ATAN2 is the two-argument variant of arctangent. Its expression is given by the following definitions:

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0 \\ \pi + \arctan\left(\frac{y}{x}\right) & y \geq 0, x < 0 \\ -\pi + \arctan\left(\frac{y}{x}\right) & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

E3/E+ does just what its name implies: adds one to the result of dividing the argument in x by one-thousand. Extensively used throughout this module and in countless matrix programs, to prepare the element indexes.

FLOOR and **CEIL**. The floor and ceiling functions map a real number to the largest previous or the smallest following integer, respectively. More precisely, floor(x) = [x] is the largest integer not greater than x and ceiling(x) =]x[is the smallest integer not less than x.

The SandMath implementation uses the native MOD function, through the expressions:
CEIL (x) = [x – MOD(x, 1)]; and FLOOR (x) = [x – MOD(x,-1)].

GEU is a new constant added to the HP-41: the Euler-Mascheroni constant, defined as the limiting difference between the harmonic series and the natural logarithm:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right)$$

The numerical value of this constant to 10 decimal places is: γ = **0.5772156649...** The stack lift is enabled, allowing for normal RPN-style calculations. It appears in formulas to calculate the Ψ (Psi) function (Digamma) and the Bessel functions.

LOGYX is the base-b Logarithm, defined by the expression:

$$\log_b(x) = \frac{\log_{10}(x)}{\log_{10}(b)} = \frac{\log_e(x)}{\log_e(b)}.$$

where the base b is expected to be in register Y, and the argument in register X.

QREM Calculates the Remainder "R" and the Quotient "Q" of the Euclidean division between the numbers in the Y (dividend) and X (divisor) registers. Q is returned to the Y registers and R is placed in the X register. The general equation is: Y = Q X + R, where both Q and R are integers.

CBRT calculates the cubic root of a number. Note that this is identical to the mainframe function **X^Y** with Y=1/3 for positive values of X, but unfortunately that results in DATA ERROR when X<0 – and therefore the need for a new function. Obviously CBRT(-x) = - CBRT(x), for x>0

Y^1/X and **X^3** are purely shortcut functions, obviously equivalent to **1/X**, **Y^X**, and to **X^2**, **LASTx**, ***** respectively - but with additional precision due to the 13-digit intermediate calculations.

Y^^X is used to calculate powers exceeding the numeric range of the calculator, simply returning the base in X and the exponent in Y. The result is shown in ALPHA in RUN mode.- For instance calculate 85^69 to obtain:

YX^ is a modified form of the native **Y^X** function, with the only difference being its tolerance to the 0^0 case – which results in DATA ERROR with the standard function but here returns 1. This has practical applications in FOCAL programs where the all-zero case is just to be ignored and not the cause for an error.

2.1.2. Number Displaying and Coordinate Conversions.

A basic set of base conversions and diverse number displaying functions round up the elementary set:

	Function	Author	Description
	ΣDGT	*Ángel Martin*	Sum of Mantissa digits
	AINT	*Frits Ferwerda*	A fixture: appends integer part of X to ALPHA
	HMS/	*Tom Bruns*	HMS Division
	HMS*	*Tom Bruns*	HMS Multiplication
	MANTEXP	*David Yerka*	Mantissa and Exponent of number
[*]	**P>R**	*Tom Bruns*	Modified Polar to Rectangular, <) in [0, 360[
[*]	**R>P**	*Tom Bruns*	Modified Rectangular to Polar, <) in [0, 360[
[*]	**R>S**	*Ángel Martin*	Rectangular to Spherical
[*]	**S>R**	*Ángel Martin*	Spherical to Rectangular
[ΣF]	**VMANT**	*Ken Emery*	Shows full-precision mantissa

AINT elegantly solves the classic dilemma to append an index value to ALPHA without its radix and decimal part - eliminating the need for FIX 0, and CF 29 instructions, taking extra steps and losing the original calculator settings. Note that HP added **AIP** to the Advantage module, and the CCD has **ARCLI** to do exactly the same.

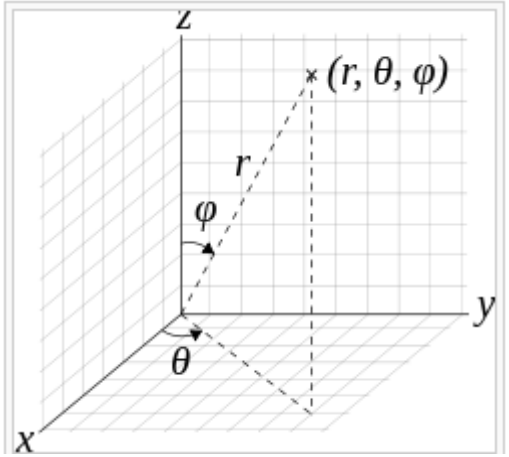
MANTEXP and **VMANT** are related functions that deal with the mantissa and exponent parts of a number. **MANTEXP** places the mantissa in X and the exponent in Y, whereas **VMANT** shows the full mantissa for a few instants before returning to the normal display form - *or permanently if any key is pressed and held during such time interval*, similar to the HP-42S implementation of "**SHOW**".

R>P and **P>R** are modified versions of the mainframe functions **R-P** and **P-R**. The difference lies in the convention used for the arguments in Polar form, which here varies between 0 and 360, as opposed to the –180, 180 convention in the mainframe.

Continuing with the coordinate conversion, **R>S** and **S>R** can be used to change between rectangular and spherical coordinates.

The convention used is shown in the figure below, defining the origin and direction of the azimuth and polar angles as referred to the rectangular axis

The SandMath implementation makes use of the fact that appropriate dual P-R conversions are equivalent to Spherical, and vice-versa.



HMS* and **HMS/** complement the arithmetic handling of numbers in HMS format, adding to the native **HMS+** and **HMS-** pair. As it's expected, the result is also put in HMS format as well.

ΣDGT is a small divertiment useful in pseudo-random numbers generation. It simply returns the sum of the mantissa digits of the argument – at light-blazing speed using just a few MCODE instructions.

More about random numbers will be covered in the Probability/Stats section later on.

Entering the base conversion section - The following functions are available in the SandMath:

	Function	Author	Description
	BS>D	*George Eldridge*	Base to Decimal
	D>BS	*George Eldridge*	Decimal to base in Y
[*]	**D>H**	*William Graham*	Decimal to Hex
[*]	**H>D**	*William Graham*	Hex to Decimal
[*]	**T>BS _ _**	*Ken Emery*	Base-Ten to Base, prompting version.

The first two are FOCAL programs, taken from the PPC ROM. They are the generic base-b to/from Decimal conversions. The Direct conversion **D>BS** expects the base in Y and the decimal number in X, returning the base-b result in Alpha. The inverse function **BS>D** uses the string in Alpha and the base in X as arguments. You can chain them to end with the same decimal number after the two executions.

T>BS (Ten to Base) is the MCODE equivalent to **D>BS**, much faster and more elegant due to its prompt – where in RUN mode you input the destination base. The result is show in the display and also left in ALPHA, so it could be also used by **BS>D** (once the base is in X). Note that the original argument (decimal value) is left in X unaltered, so you can use **T>BS** repeated times changing the base to see the results in multiple bases without having to re-enter the decimal value.

T>BS is programmable. In PRGM mode the prompt is ignored and the base is expected to be in the Y register, much the same as its FOCAL counterpart **D>BS**. Obviously using zero or one for the base will result in *DATA ERROR*. The maximum base allowed is 36 – and the *"BASE>36"* error message will be shown it that's exceeded (note that larger bases would require characters beyond "Z").

The maximum decimal value to convert depends on the destination base, since besides the math numeric factors; it's also a function of the Alpha characters available (up to "Z") and the number of them (length) in the display (12). For b=16 the maximum is 9999 E9, or 0x91812D7D600

T>BS is an enhanced version of the original function, also included in Ken Emery's book "MCODE for Beginners". The author added the PRGM-compatible prompting, as well as some display trickery to eliminate the visual noise of the original implementation. Also provision for the case x=0 was added, trivially returning the character "0" for any base. The prompt can be filled using the two top keys as shortcuts, from 1 to 10 (A-J), or the numeric keys 0-9.

Because of its importance, the hexadecimal conversions have the dedicated MCODE functions **D>H** and **H>D**. Use them to convert the number in X to its Hex value in Alpha, and vice-versa. The maximum number allowed is 0x2540BE3FF or 9,99999999 E9 decimal - much smaller than with **T>BS**, so there's a price to pay for convenience. These functions were written by William Graham and published in PPCJ V12N6 p19, enhancing in turn the initial versions first published by Derek Amos in PPCCJ V12N1 p3.

2.1.3. First, Second and Third degree Equations.

A MCODE implementation of these offers no doubt the ultimate solution, even if it doesn't involve any high level math or sophisticated technique. The Stack is used for the coefficients as input, and for the roots as output. No data registers are used.

	Function	Author	Description
	STLINE	*Ángel Martin*	Calculates straight line coefficients from two data points
[*]	**QROOT**	*Ángel Martin*	Calculates the two roots of the equation
	QROUT	*Ángel Martin*	Displays the roots in X and Y
	CROOT	*Ángel Martin*	Calculates the three roots of the equation
	CVIETA	*Ángel Martin*	Driver program for CROOT

STLINE is a simple function to calculate the straight line coefficients from two of its data points, $P_1(x1,y1)$ and $P_2(x2,y2)$. The formulas used are:

$\mathbf{Y = ax + b}$, with: $\mathbf{a = (y_2 - y_1)/(x_2 - x_1)}$, and $\mathbf{b = y_1 - a\ x_1}$

It is trivial to obtain the root once a and b are known, using: $\mathbf{x_0 = -b/a}$

Example: Get the equation of the line passing through the points (1,2) and (-1,3)

3, ENTER^, -1, ENTER^, 2, ENTER^, 1, **STLINE** -> Y: 2,500; X: -0,500

and to obtain its root: /, CHS -> X: 5,000

(*) will be shown in RUN mode *only*

For the second and third degree equations use functions **QROOT** and **CROOT**. The general forms are:

$ax^2 + bx + c = 0,$ with a#0 .

Given the quadratic equation above, **QROOT** calculates its two solutions (or roots). Input the coefficients into the stack registers: Z, Y, X using: a, ENTER^, b, ENTER^, c

The roots are obtained using the well-known formula: X1,2 = -b/2a +- sqrt[(-b/2a)^2 – c/a]
Upon execution, x1 will be left in Y and x2 will be left in X.

If the argument of the square root is negative, then the roots z_1 and z_2 are complex and conjugated (symmetrical over the X axis), with Real and Imaginary parts defined by:

Re(Z) = -b/2a
Im(Z) = sqrt[abs((-b/2a)^2 –c/a)]

z_1 = Re(z) + i Im(z)
z_2 = Re(z) – i Im(z)

Upon execution, Im(z) will be left in Y and Re(z) will be left in X.

$ax^3 + bx^2 + cx + d = 0.$ with a#0

For the cubic equation case, input the four coefficients in the stack registers T, Z, Y, X using:
a, ENTER^, b, ENTER^, c, ENTER^, d, ENTER^

CROOT uses the well-known Cardano-Vieta formulas to obtain the roots. The highest order coefficient doesn't need to be equal to 1, but errors will occur if the first term is zero (for obvious reasons).

The SandMath implementation does reasonably well with multiple roots, but sure enough you can find corner-cases that will make it fail - yet not more so than an equivalent FOCAL program. Appendix 2 lists the code, as well as an equivalent FOCAL program to compare the sizes (much shorter, but surely much slower and with data registers requirements

Both cases can return real or complex roots. If the roots are complex, the functions will flag it in the following manners:

1. **QROOT** will clear the Z register, indicating that X and Y contain the real and imaginary parts of the two solutions. Conversely, if Z#0 then X and & contain the two real roots.

2. **CROOT** will leave the calculator in RAD mode in **CROOT**, indicating that X and Y contain the real and imaginary parts of the second and third roots. The real root will always be placed in the Z register. Conversely, if the calculator is set in DEG mode then registers Z,Y, and X have the three real roots.

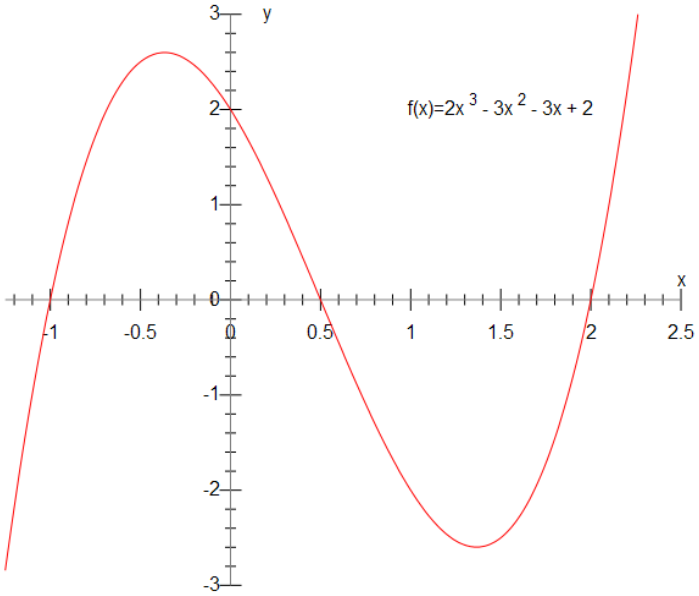
QROUT outputs the contents of the X and Y registers to the display, interpreted by the value in Z to determine whether there are tow real roots or the Real & Imaginary parts of the complex roots. It will be automatically invoked by **QROOT** (always) and by **CROOT** (real roots) when they are executed in RUN mode. Note that **CROOT** will not display the (first) real root, which will be located in Z.

CVIETA is a driver program for **CROOT**, including the prompts for the equation coefficients. The results are placed in the stack, following the same conventions as for CROOT explained above.

Example 1:- Calculate the roots of the equation: $f(x) = 2x^3 - 3x^2 - 3x + 2$.

2, [ENTER^], -3, [ENTER^], [ENTER^], 2, [**CROOT**] -> Z: 0,500; Y: -1,000; X: 2,000

From the final prompt you know all roots are real. The value in Z blinks briefly in the display before the final prompt above is presented; use RCL Z (or RDN, RDN) to retrieve it. No user registers are used.



Example 2: Calculate the three solutions of the equation: $\mathbf{x^3 + x^2 + x + 1 = 0}$

1, [ENTER^], [ENTER^], [ENTER^], [**CROOT**] → Z: -1,000; Y: 1,000; X: 1 E-10

, Shown as rounded number for the real part.

Appendix 2.- CVIETA equivalent FOCAL program, replaced now with an all-MCODE implementation.

01	LBL "CVIETA"	
02	-AMC MATH	"Running" message
03	R^	
04	ST/ T (0)	a'2/a'3 in T
05	ST/ Z (1)	a'1/a'3 in Z
06	/	
07	STO 00	a0 = a'0 / a'3
08	RDN	
09	STO 01	a1 = a'1 / a'3
10	RDN	a2 = a'2 / a'3
11	3	
12	/	
13	STO 02	a2/3
14	X^3	a2^3/27
15	ST+ X (3)	2*a2^3/27
16	RCL 01	a1
17	RCL 02	a2/3
18	*	a1*a2/3
19	-	2*a2^3/27 - a1*a2/3
20	RCL+ (00)	Showing off... :-)
21	2	
22	/	
23	STO 03	a0/2 + a2^3/27 - a1*a2/6
24	X^2	(a0/2 + a2^3/27 - a1*a2/6)^2
25	RCL 01	a1
26	RCL 02	a2/3
27	X^2	a2^2/9
28	3	
29	*	a2^2/3
30	-	a1-a2^2/3
31	STO 01	a1-a2^2/3
32	3	
33	/	1/3 (a1 - a2^2/3)
34	X^3	1/27 (a1 - a2^2/3)^3
35	+	1/27 (a1 - a2^2/3)^3 + (a0/2 + a2^3/27 - a1*a2/6
36	X<=0?	
37	GTO 01	yes, all real roots
38	SQRT	complex roots
39	ENTER^	
40	ENTER^	RPLX
41	RCL 03	a0/2 + a2^3/27 - a1*a2/6
42	-	
43	CBRT	
44	STO 01	cbrt(+x-R3/2)
45	X<>Y	
46	CHS	
47	RCL 03	a0/2 + a2^3/27 - a1*a2/6
48	-	
49	CBRT	
50	STO 03	cbrt(-x-R3/2)
51	+	
52	RCL 02	a2/3
53	-	
54	"X1"	
55	ARCL X (3)	
56	AVIEW	
57	STO 00	real root
58	RCL 01	cbrt(+x-R3/2)
59	RCL 03	cbrt(-x-R3/2)
60	-	
61	3	
62	SQRT	
63	*	
64	2	
65	/	imaginary part
66	RCL 01	cbrt(+x-R3/2)
67	RCL 03	cbrt(-x-R3/2)
68	+	
69	2	
70	/	
71	CHS	
72	RCL 02	a2/3
73	-	real part
74	,	
75	STO T (0)	flag it as Complex
76	RDN	Z=0 indicates it
77	QROUT	
78	STO 01	
79	X<>Y	
80	STO 02	
81	RTN	
82	LBL 01	all real roots
83	DEG	
84	LASTX	
85	CHS	
86	SQRT	
87	ST+ X (3)	
88	X#0?	
89	1/X	
90	RCL 03	a0/2 + a2^3/27 - a1*a2/6
91	ST+ X (3)	a0 + 2*a2^3/27 - a1*a2/3
92	CHS	
93	*	
94	ACOS	
95	3	
96	/	
97	STO 03	
98	LASTX	
99	E3/E+	
100	STO 05	1,003
101	RCL 01	a1-a2^2/3
102	3	
103	/	a1/3-a2^2/9
104	CHS	a2^2/9 - a1/3
105	SQRT	
106	ST+ X (3)	
107	STO 04	2*SQR(a2^2/9 - a1/3)
108	LBL 08	
109	RCL 03	
110	COS	
111	RCL 04	
112	*	
113	RCL 02	a2/3
114	-	
115	"X"	
116	AIRCL	Alpha integer REG
117	5	05
118	"/-="	
119	ARCL X(3)	
120	AVIEW	
121	STO IND 05	
122	120	
123	ST+ 03	
124	ISG 05	
125	GTO 08	
126	END	

2.1.4. Additional Tests: Rounded and otherwise.

Ending the first section we have the following additional test functions:

	Function	Author	Description
[*]	**X=1?**	*Nelson C. Crowle*	Is X (exactly) equal to 1?
[*]	**X>=Y?**	*Ken Emery*	Is X equal to or greater than Y?
[*]	**X>=0?**	*Ángel Martin*	Is X equal to or greater than zero?
[*]	**X=YR?**	*Ángel Martin*	Rounded Comparison
[**F**]	**FRC?**	*Ángel Martin*	Is X a fractional number?
[**F**]	**INT?**	*Ángel Martin*	Is X an integer number

They follow the general rule, returning *YES / NO* in RUN mode, and skipping a program line if false in a program. Their criteria are self-explanatory for the first three. These functions come very handy to reduce program steps and improve the legibility of the FOCAL programs.

X>=Y? compares the values in the X and Y registers, skipping one line if false.
X>=0? compares with zero the value in the X register, skipping one line if false.

These functions are arguably "missing" on the mainframe set; a fact partially corrected with the indirect comparison functions of the CX model (**X>=NN?**), but unfortunately not quite the same.

X=1? is a quick and simple way to check whether the value in X equals one. As usual, program execution skips one step if the answer is false.

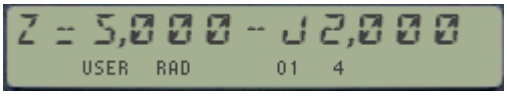
X=YR? establishes the comparison of the rounded values of both X and Y, according to the current decimal digits set in the calculator. Use it to reduce the computing time (albeit at a loss of precision) when the algorithms have slow convergence or show unstable results for larger number of decimals.

INT? and **FRC?** are two more test functions which criteria is the integer or fractional nature of the number in X. Having them available comes very handy for decision branching in FOCAL programs. The Fractions section of the module is the natural placement for them.

The remaining functions shown in the table below really are "displaced"- in that their entries are in the upper page but certainly have nothing to do with High-Level math. Call it a misdemeanour if you want, and allow me to include them now and get them out of the way.

	Function	Author	Description
	INCX	*Ken Emery*	Increases X by one
	INCY	*Ángel Martin*	Increases Y by one
	DECX	*Ángel Martin*	Decreases X by one
	DECY	*Ángel Martin*	Decreases Y by one
	ZOUT	*Ángel Martin*	Combines the values in Y and X into a complex result

Of these only **ZOUT** has been used in FOCAL programs in the SandMath, - so the others only perform a FAT entry placeholder function, and could be removed (replaced by others) in future versions of the module.



2.2. FRACTIONS	

2.2.1. Fraction Arithmetic and Displaying.

A rudimentary set of fraction arithmetic functions is included in the SandMath, including the four basic operations plus a fraction viewer and two test functions.

	Function	Author	Description
[*]	**-FCR**	*Ángel Martin*	Fractions Launcher
[**F**]	**D>F**	*Frans de Vries*	Calculates a fraction that gives the number in X
[**F**]	**F+**	*Ángel Martin*	Fraction addition
[**F**]	**F-**	*Ángel Martin*	Fraction subtraction
[**F**]	**F/**	*Ángel Martin*	Fraction multiplication
[**F**]	**F***	*Ángel Martin*	Fraction division
[**F**]	**FRC?**	*Ángel Martin*	Is X a fractional number?
[**F**]	**INT?**	*Ángel Martin*	Is X an integer number

D>F is the key function within this group. Shows in the display the *smallest possible fraction* that results in the decimal number in X, for the current display precision set. Change the display precision as appropriate to adjust the accuracy of the results.

This means the fraction obtained may be different depending on the settings, returning different results. For example, the following approximations are found for π:

π ~ 104348/33215 in FIX 9, FIX 8 and FIX 7
π ~ 355/113 in FIX 6, FIX 5 and FIX 4
π ~ 333/106 in FIX 3
π ~ 22/7 in FIX 2, FIX 1 and FIX 0

This function was written by Frans de Vries, and published in DataFile, DF V9N7 p8. It uses the same algorithm as the PPC ROM "DF" routine.

As per the fraction arithmetic functions, there's not much to say about them – apart from the fact that they use the four stack levels to enter both fractions components (the inputted values are expected to be all integers), and return the numerator and denominator of the result fraction in registers Y and X respectively. In RUN mode the execution continues to show the fraction result in ALPHA, according to the currently set number of decimals (see below).

The fraction arithmetic functions can be used in chained calculations, there's no need to re-enter the intermediate results, and the Stack enabled makes unnecessary to press ENTER^. Notice that fractions are entered using the Numerator first.

To re-calculate the fraction after changing the decimal settings just press the divide key, followed by **D>F** to re-generate the fraction values.

For example calculate 2/7 over 4/13, then add 9/17 to the result.

2, ENTER^, 7, ENTER^, 4, ENTER^, 13, **F/**, 9 ENTER^, 17, **F+** → 347/238 in FIX 6 mode.

Needless to say the fractional representation display will not be produced in PRGM mode, but it'll have a silent execution instead.

Note that the fraction math functions operate on integer numbers in the stack, returning also the numerator and denominator as integers. To get the decimal number just execute [/] to divide them.

In fact that's exactly what the functions do in RUN mode: upon completion the fraction is "converted" to a decimal number, then D>F presents the final output. That's why the display settings determine the accuracy of the conversions, even if it's not obviously seen.

This has the advantage that *the result is always reduced to the best possible fit*. For instance, when calculating 2/4 plus 18/24 in program mode – with the four values in the stack – the result will be 120 in Y and 96 in X (thus 120/96). However on RUN mode (or SST'ing the program) will show the reduced fraction:

If you want to see the reduced result from a program execution you'll need to add program steps to perform the division and add a conversion to fraction after the fraction-math operation step. The code snippet below describes this (see lines 10 and 11):

```
01 *LBL "TEST"
02  2
03  ENTER^
04  4
05  ENTER^
06  18
07  ENTER^
08  24
09  F+
10  /
11  D>F
12  END
```

INT? and **FRC?** are two more test functions which criteria is the integer or fractional nature of the number in X. Having them available comes very handy for decision branching in FOCAL programs. The Fractions section of the module is the natural placement for them.

The answer is YES / NO depending on whether the condition is true or false. In program mode the following line is skipped it the test is false.

Note: Make sure the revision "G" (or higher) of the Library#4 module is installed.

2.3. HYPERBOLICS.	-HYP _ USER RAD 012 4

2.3.1. Hyperbolic Functions.

Yes there are many unanswered questions in the universe, but certainly one of them is why, oh why, didn't HP-MotherGoose provide a decent set of hyperbolic functions in the (otherwise pathetic) MATH-PAC, and worse yet -adding insult to injury- how come that error wasn't corrected in the Advantage ROM?

For sure we'll never know, so it's about time we move on and get on with our lives – whilst correcting this forever and ever. The first incarnation of these functions came in the AECROM module; I believe programmed by Nelson C. Crowle, a real genius behind such ground-breaking module - but it was also somehow limited to 10-digit precision. The versions in the SandMath all use internally13-digit routines.

	Function	Author	Description
[*]	**-HYP**	*Ángel Martin*	Hyperbolic Launcher
[H]	**HSIN**	*Ángel Martin*	Hyperbolic Sine
[H]	**HCOS**	*Ángel Martin*	Hyperbolic Cosine
[H]	**HTAN**	*JM Baillard*	Hyperbolic Tangent
[H]	**HASIN**	*Ángel Martin*	Inverse Hyperbolic Sine
[H]	**HACOS**	*Ángel Martin*	Inverse Hyperbolic Cosine
[H]	**HATAN**	*JM Baillard*	Inverse Hyperbolic Tangent

The use of the function launcher permits convenient access to these six functions without having to assign them to any key in USER mode. Efficient usage of the keyboard, which can double up for other launchers or the standard USER mode assignment if that's also required. Combining the **ΣFL** and the SHIFT keys does the trick in a clean and logical way.

and inverses:

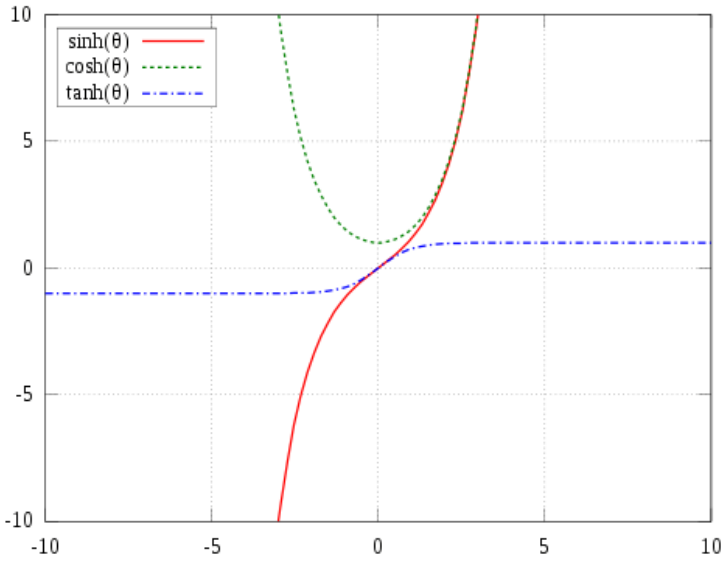
The formulas used are well known and don't require any special consideration to program.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$$

The SINH code is also used as a subroutine for the Digamma function.



The direct functions are basically exponentials, whilst the inverses are basically logarithms.

Both cases are well covered with the mainframe internal math routines without any need to worry about singularities or special error handling.

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right); x \geq 1$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}; |x| < 1$$

For all hyperbolic functions the input value is expected in X, and the return value will also be left in X. The original argument is saved in LASTx. No data registers are used.

Examples:

Complete the table below, calculating the inverses of the results to compare them with the original arguments. Use FIX 9 to see the complete decimal range.

HMKEYS assigns **-HYP** to the [SHIFT] key for convenience

x	HSIN	HASIN	HCOS	HACOS	HTAN	HATAN
1	1,175201194	1,000000000	1,543080635	1,000000000	0,761594156	0,761594156
1,001	1,176744862	1,001000000	1,544256608	1,001000000	0,762013811	1,001000000
0.01	0,010000167	0,010000000	1,000050000	0,009999958	0,009999667	0,010000000
0.0001	0,000100000	0,000100000	1,000000005	0,000100000	0,000100000	0,000100000
10	11013,23287	10,00000000	11013,23292	10,00000000	0,999999996	10,00271302

By now you've become an expert in the HYP launcher and for sure appreciate its compactness – lots of keystrokes!

With a couple of exceptions it's a100% accuracy – and really the only sore point is in the point 0.001 for the HACOS. But don't worry, there's no bugs creating havoc here – it's just the nature of the beast, bound to occur with the limited precision used (even 13-digits) in the Coconut CPU.

No wonder you're going to repeat the same table for the trigonometric functions and see how it stacks up, right?

While you're at it, go ahead and calculate the power of two of the square root, pressing:

FIX 9 , 2, SQRT , X^2 , but don't call HP to report a bug!

For very small arguments the accuracy of **SINH** and **COSH** will also start to show incorrect digits. However **HTAN** (and **HATAN**) use an enhanced formula that will hold the accuracy regardless of how small the argument is.

Note: Make sure the revision "G" (or higher) of the Library#4 module is installed.

2.4. RCL MATH.

The SandMath Module includes a set of functions written to extend the native RCL functionality – mainly in the direct math operations missing when compared to the STO equivalents, but also increasing its versatility and ease of use. There are five new RCL Math functions, plus a launcher to access them in a convenient and useful way:

	Function	Author	Description
[*]	**RCL" _ _**	*Ángel Martin*	RCL Math Launcher
[RC]	**RCL+ _ _**	*Ángel Martin*	RCL Plus
[RC]	**RCL- _ _**	*Ángel Martin*	RCL Minus
[RC]	**RCL* _ _**	*Ángel Martin*	RCL Multiply
[RC]	**RCL/ _ _**	*Ángel Martin*	RCL Division
[RC]	**RCL^ _ _**	*Ángel Martin*	RCL Power
[RC]	**AIRCL _ _**	*Ángel Martin*	ARCL integer Part of number in Register nn

2.4.1. Individual Recall Math functions.

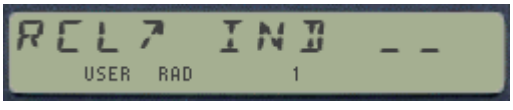
The five RCL Math new functions cover the range of four arithmetic operations (like STO does) plus a new one added for completion sake. The functions would recall the number in the register specified by the prompt, and will perform the math using the number in register X as first argument and the recalled number as the second argument.

Design criteria for these were:

1. should be prompting functions
2. should support indirect addressing (SHIFT)
3. should utilize the top 2 rows for index entry shortcut

The first condition is easy to implement in RUN mode, as it's just a matter of selecting the appropriate prompting bits in the function MCODE name. But gets very tricky when used under program mode. This has been elegantly resolved using a method first used by Doug Wilder, by means of using the program line following the instruction as the index argument. Somewhat similar to the way the HEPAX implemented it, although here there's some advantages in that the length of the index argument doesn't need to be fixed, dropping leading zeroes and even omitting it altogether if it's zero (assuming the following line isn't a numeric one which could be misinterpreted).

The indirect addressing is actually quite simple, as it simply consists of an offset added to the register number in the index. All the function code must do is remove it from the entry data provided by the OS, and the task is done. The offset value is hex 80, or 128 decimal. We'll revisit this when discussing the RCL launcher.



And the third objective is provided "for free" by the OS as well, no need for extra code at all – just using the appropriate prompting bits in the function's name.

Stack arguments are more involved than the indirect addressing. No attempt has been made to use the mainframe internal routines to accommodate this case, so stack prompts are excluded. Note that even if the Stack arguments are not directly allowed (controlled by the prompting bits), it is unfortunately possible to use the decimal key in an indirect register sequence; that is after pressing the SHIFT key. This won't work properly in the current design so must be avoided.

2.4.2. RCL Launcher – the Total Rekall.

The basic idea of a launcher is a function capable of calling a set of other functions. The grouping in this case will be for the five RCL Math functions described above, plus logically the standard RCL operation – inclusive its indirect registers addressing. Other enhancements include the prompt lengthener to three fields for registers over 99 (albeit this is de-facto limited to 128 as we'll see later on).

The keyboard mapping for [RCL] is as follows:

- Numeric keypad (or Top rows) to perform the standard RCL
- [SHIFT] for Indirect register addresses
- [EEX] for the prompt lengthener to three places
- Math keys (+, -, / , *, and ^) to invoke the RCL Match functions
- Back arrow to cancel out to the OS

Note that [RCL] is not programmable. This is done by design, so that *it can be used in a program to enter any of the RCL Math functions directly as a program line* (ignoring the corresponding prompt). The drawback is of course that the standard **RCL** operation won't be registered in a program; you must use the standard **RCL** function instead.

Notice also that indirect addressing is indeed supported by this scheme: just add hex 80 (that is decimal 128) to the register number you want to use as indirect register. As simple as that! So RCL+ IND 25 will be entered as the following two program lines: **RCL+**, followed by 153.

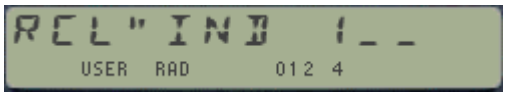
This however effectively limits the usefulness of the prompt lengthener to the range R100 to R127 – because from R128 and on *the index is interpreted as an indirect register address instead*. However, the function will allow pressing SHIF and EEX, for **a combination of IND and prompt lengthene**r which will work as expected provided that the 128 limit isn't reached – enough to make your head spin a little bit!?

Example: Store 5 in register R101, and 55555,000 in register R5.

This requires some indirect addressing as well; say using register Y the sequence would be:

101, ENTER^, 5, STO IND **Y**, and then: 55555, STO 5

Then execute RCL" IND 101 (press RCL", SHIFT, EEX, 0, 1)--> to obtain 55555,00 in X

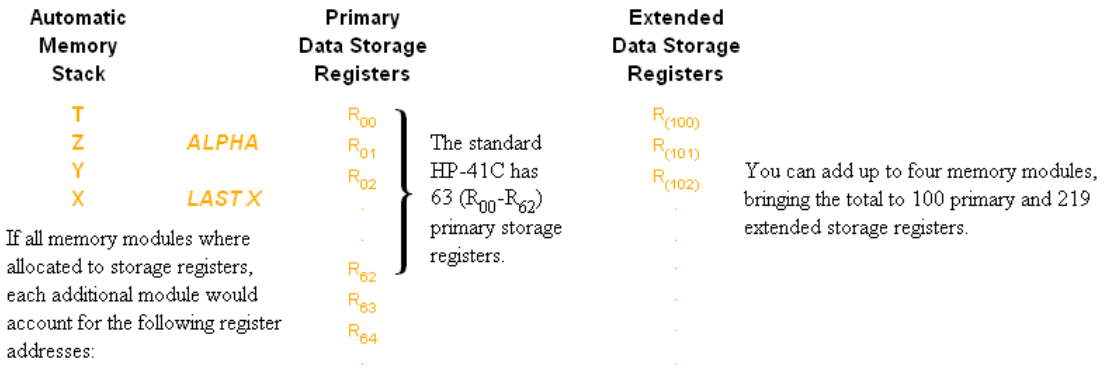


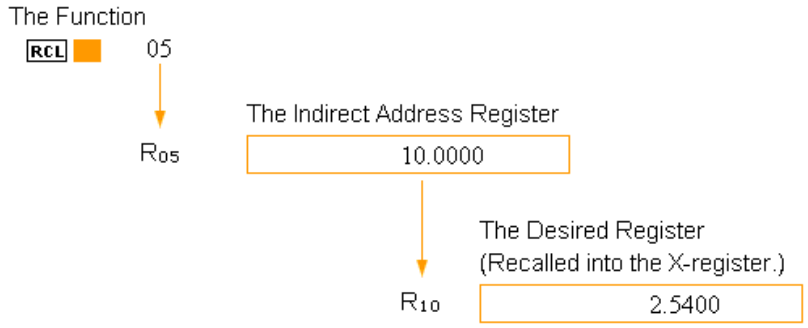
Note: general-purpose prompt lengtheners are a better alternative to the [EEX] implementation used here. Their advantage of course is that they are applicable to all mainframe prompting functions, not only to the enhanced RCL. Thus for instance, you could use it with **STO** as well, removing the need for indirect addressing to store 5 in R101. The AMC_OS/X module has a general-purpose prompt lengthener, activated by pressing the [ON] key while the function prompt is up.

Pressing [ALPHA] at the RCL prompt will invoke function **AIRCL _ _**. This will in turn prompt for a data register number, and once filled it'll append the integer part of the value stored in that register to the ALPHA register – thus equivalent to what **AINT** does with the x register.

Note that **AIRCL _ _** is fully programmable. When entered in a program you'd ignore the prompts, and the program step following it will be used to hold the register number to be used by **ARCLI** when the program runs. This technique is known as "*non-merged*" functions, to work-around the limitation of the OS – Too bad we can't use the Byte Table locations wasted by **eG0bEEP** and **W"** instead! This method is used in several functions of the SandMath module, like the RCL math functions just described.

Appendix 3.- A trip down to Memory Lane. From the HP-41 User's Handbook.-





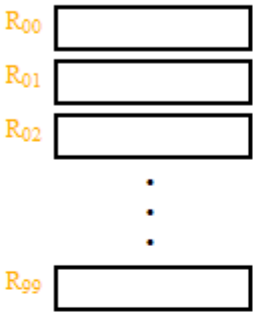
Storage Register Arithmetic

Arithmetic can be performed upon the contents of all storage registers by executing STO followed by the arithmetic function followed in turn by the register address. For example:

Opertion	Result
STO + 01	Number in X-register is added to the contents of register R_{01}, and the sum is placed into R_{01}. The display execution form of this is ST+.
STO − 02	Number in X-register is subtracted from the contents of register R_{02}, and the difference is placed into R_{02}. The display execution form of this is ST−.
STO × 03	Number in X-register is multiplied by the contents of register R_{03}, and the product is placed into R_{03}. The display execution form of this is ST×.
STO ÷ 04	Number in R_{04} is divided by the number in the X-register, and the quotient is placed into R_{04}. The display execution form of this is ST÷.

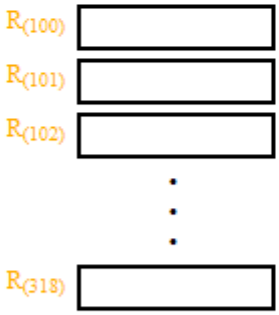
Primary Data Storage Registers

The standard HP-41C has 63 registers that can be allocated to data storage or program memmory in *any* combination. As you add HP memory Modules (up to four), the total number of registers can increase to 319-64 registers for each memory module. When allocated, data storage registers numbered R_{00} through R_{99} are Primary Data Storage Registers.



Extended Data Storage Registers

When allocated, data storage registers numbered $R_{(100)}$ through $R_{(319)}$ are extended Data Storage Registers.



Program Memory

All registers that are not allocated as Primary of Extended data storage registers are part of program memory. When allocated as program memory, the standard HP-41C registers provide space for 200-400 fully-merged lines of programs. When allocated as program memory, each memory module adds 200-400 lines. The total can be 1000-2000 lines when alle 319 registers are allocated as program memory. Variations in storage capacity depend on the kinds of functions stored in program memory.

Note: Make sure the revision "G" (or higher) of the Library#4 module is installed.

3. Upper-Page Functions in detail.

It's time now to move on to the second page within the SandMath – holding the Special Functions and the Statistical and Probability groups. Let's see first the Statistical section – easier to handle and of much less extension; and later on we'll move into high-level math, taking advantage of the extended launchers and additional functionality described in the introduction of this manual.

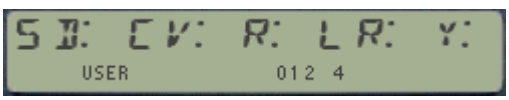
The following functions are in this general group: Some of them are plain catch-up, with the aim to complete the set of basic functions. Some others are a little more advanced, reaching into the high level math as well.

	Function	Author	Description
[*]	**%T**	*Ángel Martin*	Compound Percent of x over y
	DSP?	*Ángel Martin*	Number of decimal places
	EVEN?	*Ángel Martin*	Tests whether x is an even number
[*]	**GCD**	*Ángel Martin*	Greatest Common Divider
[*]	**LCM**	*Ángel Martin*	Least Common Multiple
	MFCT	*JM Baillard*	Multifactorial
	NCR	*Ángel Martin*	Combinations of N elements taken in groups of R
[*]	**NPR**	*Ángel Martin*	Permutations of N elements taken in groups of R
[*]	**ODD?**	*Ángel Martin*	Tests whether x in an odd number
	PDF	*Ángel Martin*	Normal Probability Density Function
	PFCT	*Ángel Martin*	Prime Factorization
	PRIME?	*Jason DeLooze*	Primality Test – finds one factor
[*]	**RAND**	*Håkan Thörgren*	Random Number from Seed (in buffer)
[*]	**RGMAX**	*JM Baillard*	Maximum in a register block
	RGSORT	*Hajo David*	Sorts a block of registers
	SEEDT	*Håkan Thörgren*	SEED with Timer
[*]	**ST<>Σ**	*Nelson C. Crowle*	ΣREG exchange with Stack
	STSORT	*David Phillips*	Stack Sort

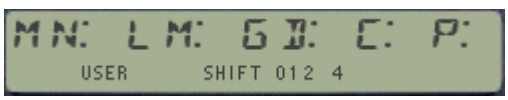
Statistical Menu - Another type of Launcher.

Pressing [ΣFL] twice will present the STAT/PROB functions menu, allowing access to 10 functions using the top row keys [A]-[J]. Two line-ups are available, toggled by the [SHIFT] key:

[ΣΣ] Default Lineup: Linear Regression



[ΣΣ] Shifted Lineup: Probability



Note the inclusion of the mainframe functions **MEAN** and **SDEV** in the menus, for a more rounded coverage of the statistical scope. With the manus up you just select the functions by pressing the key under the function abbreviated name. Use [SHIFT] to toggle back and forth between both lineups, and the back arrow key to cancel out to the OS.

Obviously the data pairs must be already in the ΣREG registers for these functions to operate meaningfully.

Alea jacta est…

It's a little known fact that the SandMath module also uses a buffer to store the current seed used for random number generation. The buffer id# is 9, and it is automatically created by **SEEDT** or **RAND** the first time any of them is executed; and subsequently upon start-up by the Module during the initialization steps using the polling points.

SEEDT will take the fractional part of the number in X as seed for RNG, storing it into the buffer. If x=0 then a *new seed will taken using the Time Module* – really the only real random source within the complete system.

RAND will compute a RNG using the current seed, using the same popular algorithm described in the PPC ROM - and incidentally also used in the CCD module's function RNG)

Both functions were written by Håkan Thörngren, an old-hand and MCODE expert - and published in PPC V13N4 p20

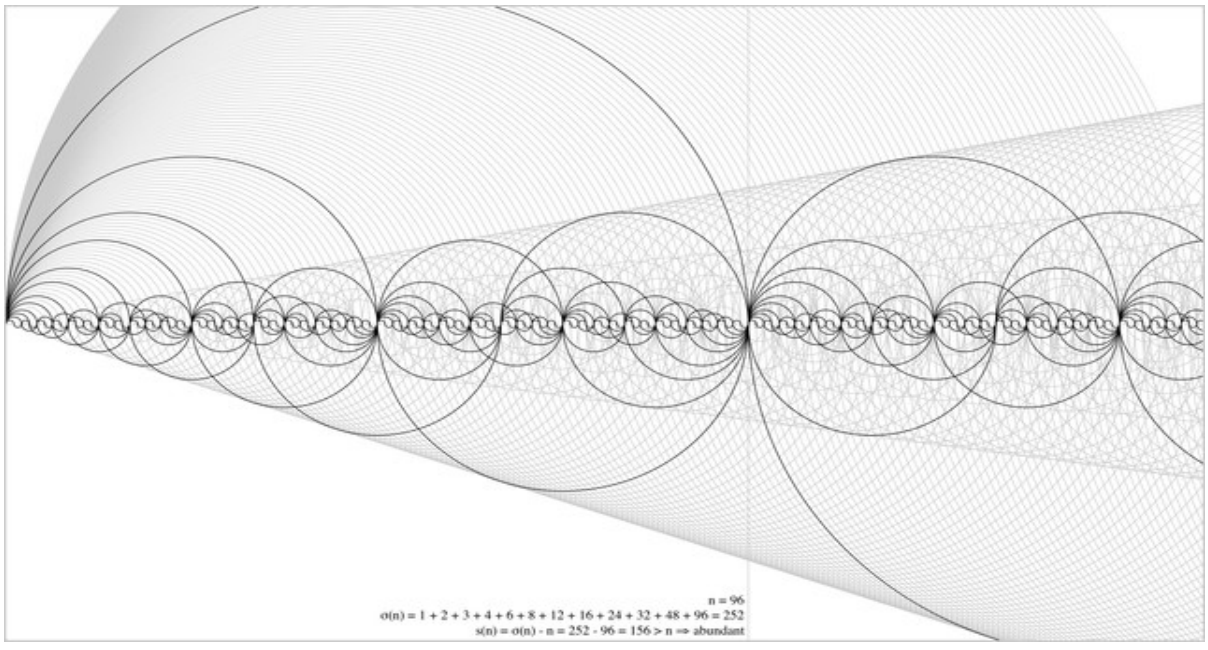
PRIME? Determines whether the number in the X register is Prime (i.e. only divisible by itself and one). If not, *it returns the smallest divisor found and stores the original number into the LASTX* register. **PRIME?** Also acts as a test: YES or NO are shown depending of the result in RUN mode. When in a program, the execution will skip one step if the result is false (i.e. not a prime number), enabling so the conditional branching options.

This gem of a function was written by Jason DeLooze, and published in PPCCJ V11N7 p30.

Example program: **-** The following routine shows the prime numbers starting with 3, and using diverse Sandbox Math functions.

```
01  LBL "PRIMES"        05  PRIME?          09  INCX
02  3                    06  VIEW X  <yes>   10  GTO 00
03  LBL 00               07  X#Y?    <no>    11  END
04  RPLX                 08  LASTX
```

See other examples later in the manual, relative to prime factorization programs.



Combinations and Permutations – two must-have classics.

Nowadays would be unconceivable to release a calculator without this pair in the function set – but back in 1979 when the 41 was designed things were a little different. So here there are, finally and for the record.

NPR calculates Permutations, defined as the number of possible different arrangements of N different items taken in quantities of R items at a time. No item occurs more than once in an arrangement, and different orders of the same R items in an arrangement are counted separately. The formula is:

$$\frac{n!}{(n-k)!}.$$

NCR calculates Combinations, defined as the number of possible sets or N different items taken in quantities or R items at a time. No item occurs more than once in a set, and different orders of the same R items is a set are not counted separately. The formula is:

$$\frac{n!}{k!(n-k)!}$$

The general operation include the following enhanced features:

- Gets the integer part of the input values, forcing them to be positive.
- Checks that neither one is Zero, and that n>r
- Uses the minimum of {r, (n-r)} to expedite the calculation time
- Checks the Out of Range condition at every multiplication, so if it occurs its determined as soon as possible
- The chain of multiplication proceeds right-to-left, with the largest quotients first.
- The algorithm works within the numeric range of the 41. Example: nCr(335,167) is calculated without problems.
- It doesn't perform any rounding on the results. Partial divisions are done to calculate **NCR**, as opposed to calculating first **NPR** and dividing it by r!

Provision is made for those cases where n=0 and r=0, returning zero and one as results respectively. This avoids DATA ERROR situations in running programs, and is consistent with the functions definitions for those singularities.

Note as well that there is no final rounding made to the result. This was the subject of heated debates in the HP Museum forum, with some good arguments for a final rounding to ensure that the result is an integer. The SandMath implementation however does not perform such final "conditioning", as the algorithm used seems to always return an integer already. Pls. Report examples of non-conformance if you run into them.

Example: Calculate the number of sets from a sample of 335 objects taken in quantities of 167:

Type: 335, ENTER^, 167, XEQ "**NCR**" -> 3,0443587 99

Example: How many different arrangements are possible of five pictures which can be hung on the wall three at a time:

Type: 5, ENTER^, 3, XEQ "**NPR**" -> 60,00000000

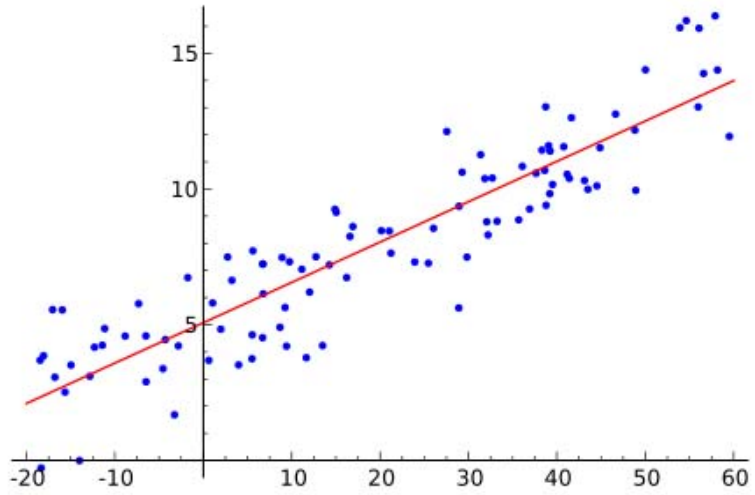
The execution time for these functions may last several seconds, depending on the magnitude of the inputs. The display will show "RUNNING…" during this time.

Linear Regression – Let's not digress.

The following four functions deal with the Linear Regression, the simplest type of the curve fitting approximations for a set of data points. They complement the native set, which basically consists of just **MEAN** and **SDEV**.

	Function	Author	Description
[ΣΣ]	**CORR**	*JM Baillard*	Correlation Coefficient of an X,Y sample
[ΣΣ]	**COV**	*JM Baillard*	Covariance of an X,Y sample
[ΣΣ]	**LR**	*JM Baillard*	Linear Regression of an X,Y sample
[ΣΣ]	**LRY**	*JM Baillard*	Y- value for an X point

Linear regression is a statistical method for finding a straight line that best fits a set of two or more data pairs, thus providing a relationship between two variables. By the method of least squares, **LR** will calculate the slope A and Y-intercept B of the linear equation: Y = Ax + B.



The results are placed in Y and X registers respectively. When executed in RUN mode the display will show the straight-line equation, similar to the **STLINE** function described before:

COV will calculate the sample covariance. **CORR** will return the correlation coefficient, and **YLR** the linear estimate for a given x.

Example: find the y-intercept and slope of the linear approximation od the data set given below:

X	0	20	40	60	80
Y	4.63	5.78	6.61	7.21	7.78

Assuming all data pairs values have been entered using Y-value, ENTER^, X-value , Σ+; we type:

XEQ "**LR**" -> 0,038650000 and X<>Y -> 0,038650000 producing the following output in FIX 2:

Ratios, Sorting and Register Maxima.

%T is a miniature function to calculate the percent of a number relative to another one (its reference). The formula is %T(y,x) = 100 x / y

Example: the relative percent of 4 over 25 is 16%.

GCD and **LCM** are fundamental functions also inexplicably absent in the original function set. They are short and sweet, and certainly not complex to calculate. The algorithms for these functions are based on the PPC routines **GC** and **LM** – conveniently modified to get the most out of MCODE environment.

If a and b are not both zero, the greatest common divisor of a and b can be computed by using least common multiple (lcm) of a and b:

$$\gcd(a,b) = \frac{a \cdot b}{\operatorname{lcm}(a,b)}.$$

Examples: GCD(13,17) = 1 (primes), GCD(12,18) = 6; GCD(15,33) = 3
Examples: LCM (13,17) = 221; LCM(12,18) = 36; LCM(15,33) = 165

RGSORT sorts the contents of the registers specified in the control number in X, defined as: ***bbb,eee***, where "**bbb"** is the begin register number and "**eee"** is the end register number. If the control number is positive the sorting is done in ascending order, if negative it is done in descending order. This function was written by HaJo David, and published in PPCCJ V12N5 p44.

STSORT sorts in descending order the contents of the four stack registers, X, Y, Z and T. No input parameters are required. This function was written by David Phillips, and published in PPCCJV12N2 p13

RGMAX finds the maximum within a block of consecutive registers – which will be placed in X, returning also the register number to Y. The register block is defined with the control word in X as input, with the same format as before: bbb.eee. It was written by Jean-Marc Baillard.

ST<>Σ exchanges the contents of the statistical registers and the stack. Use it as a convenient method to review their values when knowing their actual location is not required.

ODD? And **EVEN?** are simple tests to see is the number in X is odd or even. The answer is YES / NO, and in program mode the following line is skipped it the test is false. The implementation is based on the MOD function, using MOD(x,2) = 0 as criteria for evenness.

INCX, **DECX**, **INCY** and **DECY** are convenient substitutes for 1,+ and 1,- These functions will decrement or increment by one the current content of the X or Y registers. Also used instead of ISG X and DSE X, when there's no desire to branch the program execution even if the boundary condition is reached: this saves a NOP line placed right after the conditional instruction.

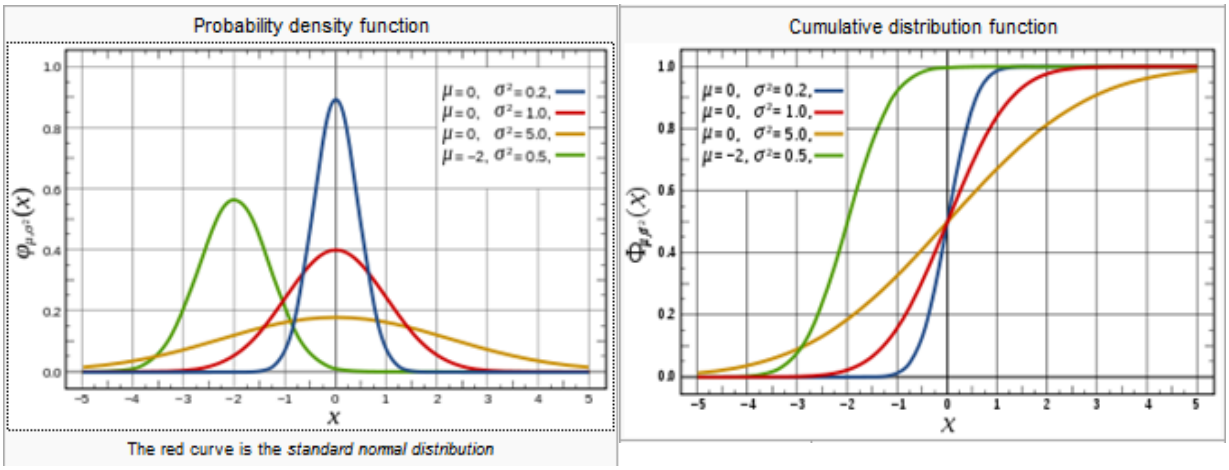
DSP? returns in X the number of decimal places currently set in the display mode 0 regardless whether it's FIX, SCI , or END. Little more than a curiosity, it can be used to restore the initial settings after changing them for displaying or formatting purposes.

(Normal) Probability Distribution Function.

In probability theory, the normal (or Gaussian) distribution is a continuous probability distribution that has a bell-shaped probability density function, known as the Gaussian function or informally as the bell curve:

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The parameter μ is the mean or expectation (location of the peak) and σ□2 is the variance. σ is known as the standard deviation. The distribution with μ = 0 and σ^2 = 1 is called the standard normal distribution or the unit normal distribution



The figure above shows both the density functions as well as the cumulative probability function for several cases. The Error function **ERF** in the SandMath can be used to calculate the **CDF** – no need to apply brute force and use **PFD** in an **INTEG**-like scenario, or course. The relation to use is:

$$F(x;\,\mu,\sigma^2) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right], \quad x \in \mathbb{R}.$$

Example program: The routine below calculates CDF. Enter μ, σ, and x values in the stack.

```
01  LBL "CDF"      08  /
02  RCL Z           09  ERF
03  -               11  INCX
04  X<>Y            12  2
05  /               13  /
06  2               14  END
07  SQRT
```

PDF expects the mean and standard deviation in the Z and Y stack registers, as well as the argument x in the X register. Upon completion x will be saved in LASTx, and f(μ,σ,x) will be placed in X. It has an all-MCODE implementation, using 13-digit routines for increased accuracy.

PDF is a function borrowed from the Curve Fitting Module, which contains others for different distribution types. With the Normal distribution being the most common one, it was the logical choice to include in the SandMath.

And what about prime factorization?

Function **PFCT** will do a very fast and simple prime factorization of the number in X, using PRIME? To look for the successive divisors until 1 is found. **PFCT** uses the ALPHA registers to present the results, grouping the repetitions of a given factor in its corresponding exponent.

For example, for x=126 the three prime factors are 2, 3, and 7, with 3 repeated two times:

For large numbers or when many prime factors exist, the display will scroll left to a maximum length of 24 characters. This is sufficient for the majority of cases, and only runs into limiting situations in very few instances, if at all – remember that exceeding 24 characters will shift off the display the left characters first, that is the original number - which doesn't result into any data loss.

Obviously prime numbers don't have any other factors than themselves. For instance, for x=17777 PFCT will return:

, which indeed is hardly debatable.

Note that only the last two prime factors found will be stored in Y and Z, and that the original number will remain in X after the execution terminates. A more capable prime factorization program is available in the ALGEBRA module, using the matrix functions of the Advantage and Advanced Matrix ROMs to save the solutions in a results matrix. See the appendices for a listing of the program used in the SandMath and the more comprehensive one.

1	LBL "PRMF"
2	INT
3	ABS
4	CLA
5	AINT
6	"/-="
7	X=1?
8	GTO 01
9	CF 00
10	LBL 00
11	PRIME?
12	SF 00
13	AINT
14	FS?C 00
15	GTO 01
16	LASTX
17	X<>Y
18	/
19	"/-*"
20	GTO 00
21	LBL 01
22	X=1?
23	AINT
24	AVIEW
25	END

Shown on the left there's an even simpler version, that doesn't consolidate the multiple factors – which will aggravate the length limitation of the ALPHA registers of 24-chrs max. The core of the action is performed by **PRIME?**, therefore the fast execution due to the MCODE speed.

See the appendix in the next pages, with both the actual code for **PFCT** in the SandMath , and for **PRMF** - a more capable implementation using the Matrix functions from the HP Advantage to store the prime factor and their repetition indexes – really the best way to present the results.

For that second case the function **PF>X** restores the original argument from the matrix values. Also function **TOTNT** is but a simple extension, using the same approach.

Appendix 4. Prime factor decomposition.

The FOCAL programs listed below are for **PFCT** – included in the SandMath – and **PRMF**, a more capable implementation that uses the Matrix functions from the HP Advantage (or the AVD_MATRIX ROM).

PRMF stores all the different prime factors and their repetition indices in a (n x 2) matrix. The matrix is re-dimensioned dynamically each time a new prime factor is found, and the repetition index is incremented each time the same prime factor shows up.

A matrix is a much better place than the ALPHA register for sure – as is done in **PFCT**. The drawback is of course the execution speed, much faster in **PFCT.**

PF>X is the reverse function that restores the original number from the values stored in the matrix.

TOTNT (Totient function) is but a simple extension, also shown in the listings below.

PRMF, PC>X and **TOTNT** are included in the Advanced MATRIX ROM.

Below is the program listing implemented in the SandMath:

1		*LBL "PRMF"*		32		*GTO 03*	*skip if Prime*
2		CF 00		33		ST/ L	
3		INT		34		LASTX	*reduced number*
4		ABS		35		*GTO 00*	
5		CLA		36		*LBL 03*	*Prime found*
6		AINT		37		RCL 00	
7		"/-="		38		X=1?	
8		PRIME?		39		*GTO 01*	
9		SF 00		40		"/-^"	
10		AINT	*first prime factor*	41		AIP	
11		X=1?		42		*GTO 01*	
12		*GTO 01*		43		*LBL 02*	NEW pf
13		FS?C 00	*if prime, we're done*	44		STO 01	
14		*GTO 01*		45		RCL 00	
15		STO 01	*save for grouping*	46		X=1?	
16		ST/ L		47		*GTO 03*	
17		LASTX	*reduced number*	48		"/-^"	
18		*LBL 05*		49		AINT	
19		E		50		*LBL 03*	
20		STO 00	*reset counter*	51		RDN	
21		RDN		52		"/-*"	
22		*LBL 00*		53		AINT	
23		RCL 01	*previous prime factor*	54		FS?C 00	
24		X<>Y		55		*GTO 01*	
25		PRIME?		56		ST/ L	
26		SF 00		57		LASTX	
27		X#Y?	*different PF?*	58		*GTO 05*	
28		*GTO 02*	YES	59		*LBL 01*	
29		ISG 00	*increase counter*	60		AVIEW	
30		NOP	SAME pf	61		ANUM	
31		FS?C 00	*was it prime?*	62		END	

Below is the Enhanced version, allowing for any number of different prime factors and repetition indices – all stored in a (n x 2) matrix file in extended memory, "PRMF".

Note how the program structure is basically the same, despite the addition of the matrix handling. Since the Advantage module is required we've used **AIP** instead of **AINT**, totally interchangeable as they're basically the same function.

```
01  LBL "PRMF"
02  "PRMF"
03  2
04  E3/E+
05  MATDIM
06  CLX
07  MSIJA
08  RDN
09  CF 00
10  INT
11  ABS
12  PRIME?
13  SF 00
14  MSR+
15  X=1?
16  GTO 01
17  FS?C 00          if prime, we're done
18  GTO 01
19  STO 01           save for grouping
20  ST/ L
21  LASTX            reduced number
22  LBL 05
23  E
24  STO 00           reset counter
25  RDN
26  LBL 00
27  RCL 01           previous prime factor
28  X<>Y
29  PRIME?
30  SF 00
31  X#Y?             different PF?
32  GTO 02           YES
33  ISG 00           increase counter
34  NOP              SAME pf
35  FS?C 00          was it prime?
36  GTO 03           skip if Prime
37  ST/ L
38  LASTX
39  GTO 00
40  LBL 03
41  RCL 00
42  MSR+
43  GTO 01
44  LBL 02           NEW pf
45  STO 01
46  RCL 00
47  MSR+
48  RDN
49  ST/ L
50  LASTX
51  DIM?
52  INCX
53  MATDIM
54  X<> Z
55  MSR+
56  FS?C 00          was it prime?
57  GTO 01           yes, wrap up
58  X<>Y             no, swap things
59  GTO 05           and do next PF
60  LBL "PF>X"
61  SF 04
62  SF 10
63  "PRMF"
64  LBL 01
65  E
66  FC? 10
67  MSR+
68  STO 00           re-build the number
69  MSIJA
70  CLA
71  LBL 06
72  MRR+
73  FC? 04
74  AIP
75  MRR+
76  FC? 04
77  X=1?
78  GTO 04
79  "/-^"
80  AIP
81  LBL 04
82  Y^X
83  ST* 00
84  FC? 10
85  "/-*"
86  FC? 10
87  GTO 06
88  RCL 00
89  PROMPT
90  END
```

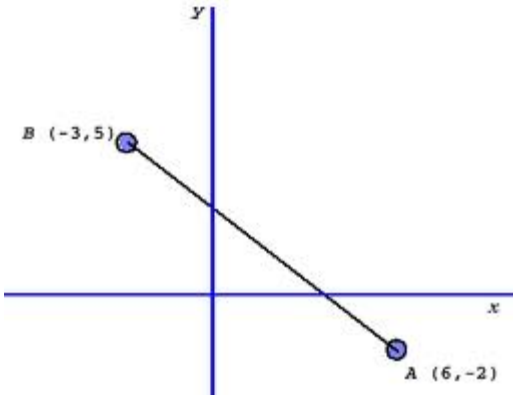
Distance between two points.

The Euclidean distance between points p and q is the length of the line segment connecting them. In the Euclidean plane, if p = (p1, p2) and q = (q1, q2) then the distance is given by

$$\mathrm{d}(\mathbf{p},\mathbf{q}) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}.$$

PP2 expects the coordinates of the two points stored in the stack, (y1,x1), (y2,x2) in T,Z,Y, and X (or vice-versa). The distance will be placed in X upon completion.

Example: Calculate the distance between the points a(-3,5) and b(6,-2) from the figure below:

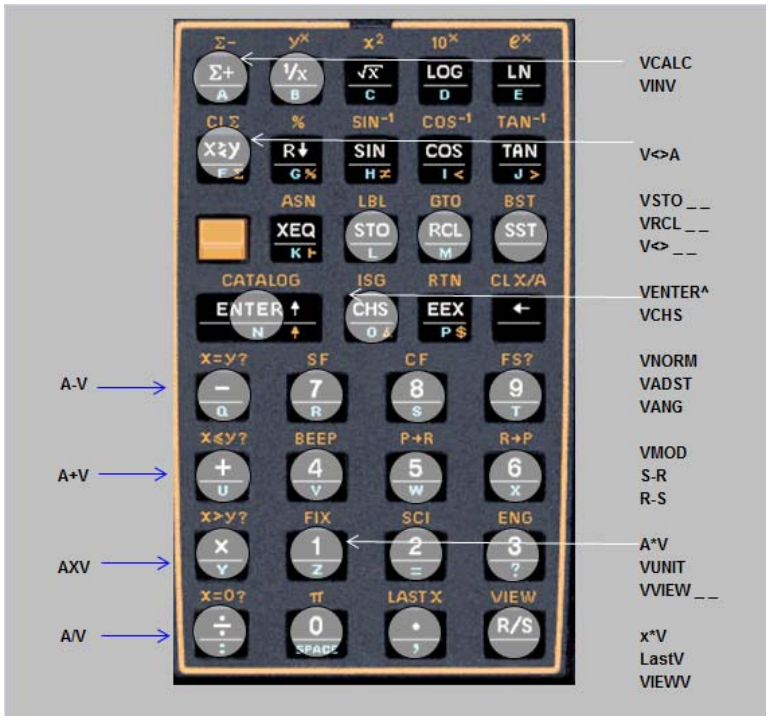


Type: 5, ENTER^, -3, ENTER^, -2, ENTER^, 6, ENTER^,

XEQ "**PP2**" -> 9,899494937

Note: A similar function exists in the 41Z module – **ZWDIST**, which basically calculates the same thing, albeit done in a complex-number context.

Note: **PP2** is the only geometry function within the SandMath. The **VECTOR ANALYSIS** module contains many more, as well as a full-featured 3D-Vector Calculator (see overlay below). It is a 4k-module that can be used independently from the SandMath, but sure it is a powerful complement for these specific subjects.



3.2 FACTORIALS

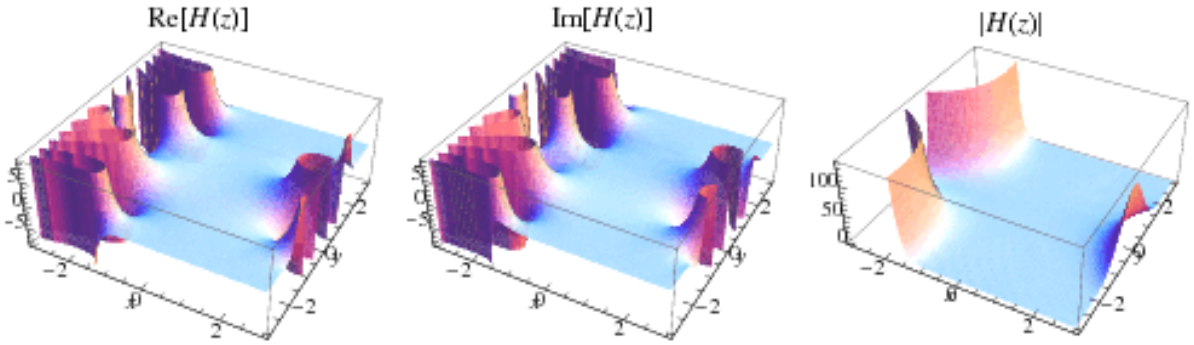
Quick recap: a summary table of the different factorial functions available in the SandMath.-

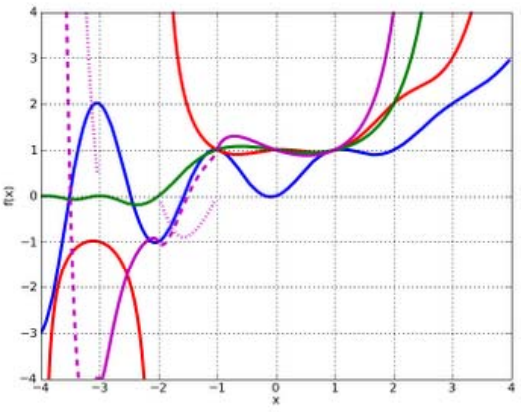
	Function	Author	Description
	MFCT	*JM Baillard*	Multifactorial
	LGMF	*Ángel Martin*	Logarithm Multifactorial
[Σ$]	**SFCT**	*JM Baillard*	Superfactorial
[Σ$]	**XFCT**	*JM Baillard*	Extended FACT
	POCH	*Ángel Martin*	Pochhammer symbol
[Σ$]	**FFACT**	*Ángel Martin*	Falling factorial

Large numbers in a calculator like the HP-41 represent a challenge. Not only the numeric range represents a problem, but also the reduced accuracy limits the practical application of the field. Nevertheless the few functions that follow contribute to add further examples of the ingenuity and what's possible using this venerable platform.

This was the last section added to the SandMath, in revision "E". It also required compacting the few gaps available, and transferring some code to the last available space in the Library#4 module. Make sure you have matching revision of those two!

The functions in the table above *operate only on integers*, i.e. no extension to real numbers using GAMMA. Below one of such extensions, the Hyperfactorial in a 3D visualization from WolframWorld:





The figure on the left shows a plot of the four functions on the real line (Fibonacci in blue, double factorial in red, superfactorial in green, hyperfactorial in purple).

Don't expect quantum leaps in number theory here; it is after all one of the most difficult branches of math.

Pochhammer symbol: Rising and falling empires.

In mathematics, the Pochhammer symbol introduced by Leo August Pochhammer is the notation $(x)^n$, where n is a non-negative integer. Depending on the context the Pochhammer symbol may represent either the rising factorial or the falling factorial as defined below. Care needs to be taken to check which interpretation is being used in any particular article.

The symbol $x^{(n)}$ is used for the rising factorial (sometimes called the "Pochhammer function", "Pochhammer polynomial", "*ascending factorial*", "rising sequential product" or "upper factorial"):

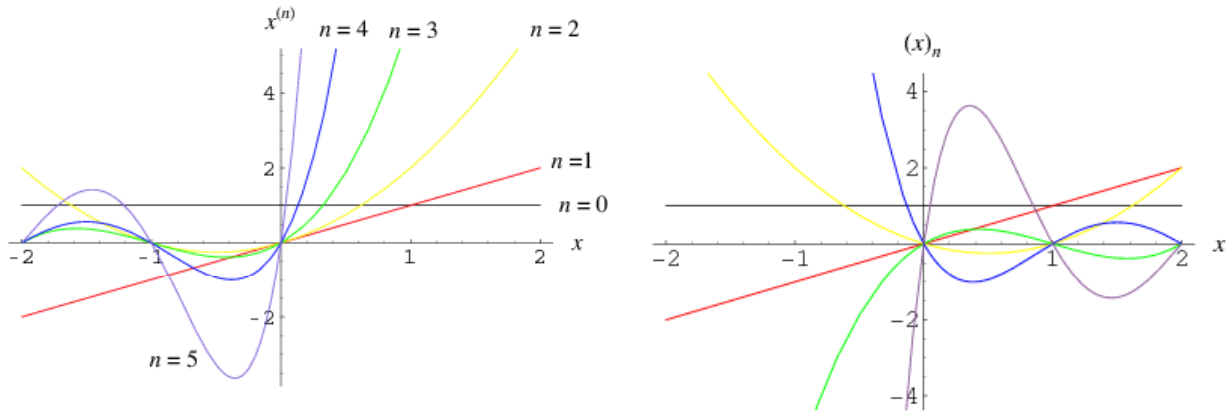
$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1).$$

The symbol $(x)_n$ is used to represent the falling factorial (sometimes called the "descending factorial",[2] "falling sequential product", "lower factorial"):

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1)$$

These conventions are used in combinatory. However in the theory of special functions (in particular the hypergeometric function) the Pochhammer symbol (x)n is used to represent the rising factorial. Extreme caution is therefore needed in interpreting the meanings of both notations !

The figures below show the rising (left) and falling factorials for n={0,1,2,3,4}, and -2<x<2



Function **POCH** calculates the rising factorial. It expects n and x to be in the Y and X registers respectively (i.e. the usual convention). For large values of n the execution time may be very long – you can hit any key to stop the execution at any time.

The falling factorial is related to it (a.k.a. Pochhammer symbol) by :

$$(x)_n = (-1)^n\,(-x)^{(n)},$$

The usual factorial n! Is related to the rising factorial by: $n\,! = 1^{(n)}$

Whereas for the falling factorial the expression is: $n\,! = (n)_n$

Examples: Calculate the rising factorial for n=7, x=4, and the falling factorial for n=7, x=7

7, ENTER^, 4, XEQ "**POCH**" -> 604.800,0000,
7, ENTER^, 7, CHS, XEQ "**POCH**", 7, XEQ "**CHSYX**" -> 5.040,000000

Multifactorial, Superfactorial and Hyperfactorial.

This section covers the main extensions and generalizations of the factorial. There are different ways to expand the definition, depending on the actual sequences of numbers used in the calculation.

The **double factorial** of a positive integer n is a generalization of the usual factorial **n!**, defined by:

$$n!! \equiv \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1 & n > 0 \text{ odd} \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2 & n > 0 \text{ even} \\ 1 & n = -1, 0. \end{cases}$$

Even though the formulas for the odd and even double factorials can be easily combined into:

$$n!! = \prod_{i;\ 0 \le 2i < n} (n - 2i),$$

The double factorial is a special case of the **multifactorial**, which uses the same formula but with different "steps": subtracting "k" (instead of "2") from the original number, thus:

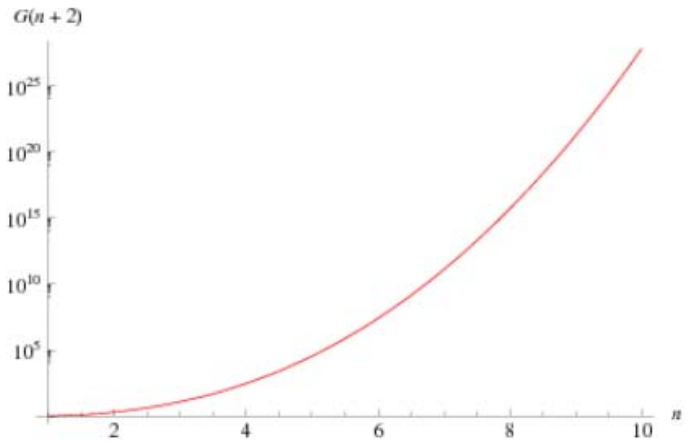
$$\begin{aligned} n! &= n\,(n-1)\,(n-2) \cdots 2 \cdot 1 \\ n!! &= n\,(n-2)\,(n-4) \cdots \\ n!!! &= n\,(n-3)\,(n-6) \cdots, \end{aligned}$$

where the products run through positive integers. Obviously for k=1 we have the standard **FACT**. One can define the k-th factorial, denoted by $\mathbf{n!}^{(k)}$ recursively for non-negative integers as:

$$n!^{(k)} = \begin{cases} 1, & \text{if } 0 \le n < k, \\ n((n-k)!^{(k)}), & \text{if } n \ge k\,, \end{cases}$$

Another extension to the factorial is the **Superfactorial**. It doesn't use any step-size as variant, rather it follows a similar formula but using the factorial of the numbers instead of the numbers themselves:

$$\mathrm{sf}(n) = \prod_{k=1}^{n} k! = \prod_{k=1}^{n} k^{n-k+1} = 1^n \cdot 2^{n-1} \cdot 3^{n-2} \cdots (n-1)^2 \cdot n^1.$$



Both the multifactorial and (specially) the superfactorial will exceed the calculator numeric range rather quickly, so the SandMath functions use a separate mantissa and exponent approach, using registers X and Y respectively.

Nevertheless the functions will put up a consolidated (combined) representation in the display, using the letter "E" to separate both amounts. Make sure to adjust the FIX settings as approriate:

Examples: Calculate the multi- and superfactorials given below:

2345 !! !! !! type: 6, ENTER^, XEQ "**MFCT**" ->

1234 !! !! ! type: 5, ENTER^, XEQ "**MFCT**" ->

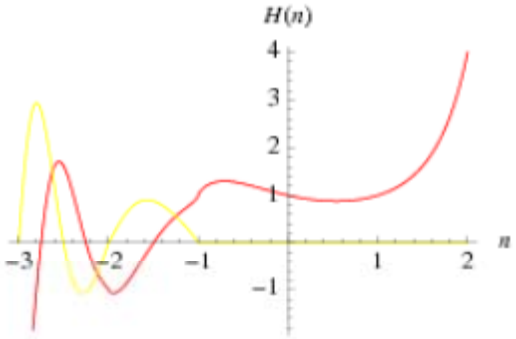
Sf(41) type: 41, **ΣFL$** "**SFCT**" ->

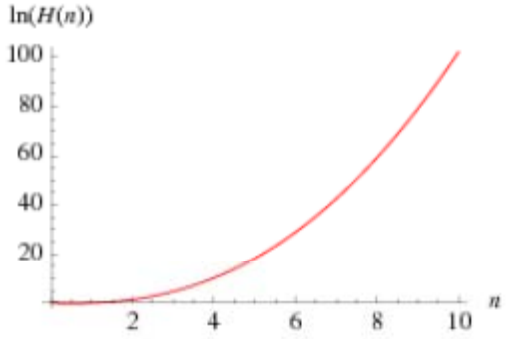
Sf(100) type: 100, **ΣFL$** "**SFCT**" ->

To complete this trinity of factorials – Occasionally the **hyperfactorial** of n is considered. It is written as H(n) and defined by:

$$H(n) = \prod_{k=1}^{n} k^k = 1^1 \cdot 2^2 \cdot 3^3 \cdots (n-1)^{n-1} \cdot n^n.$$

The figures below show a plot for both the hyperfactorial and its logarithm – itself a convenient scale change very useful to avoid numeric range problems. Note that they're extended to all real arguments, and not only the natural numbers – also called the "K-function".





See below a couple of simple FOCAL program to calculate the hyperfactorial (which runs beyond the numeric range dramatically soon!) and its logarithm written by JM Baillard. Understandably slow and limited as these programs are, you can visit his web site for a comprehensive treatment using dedicated MCODE functions for the many different possible cases.

```
01  LBL "HFCT"
02  1
03  LBL 01
04  RCL Y
05  ENTER^
06  Y^X
07  *
08  DSE Y
09  GTO 01
10  END
```

```
01  LBL "LOGHF"
02  0
03  LBL 01
04  RCL Y
05  ENTER^
06  LOG
07  *
08  +
09  DSE Y
10  GTO 01
11  END
```

Logarithm Multi-Factorial.

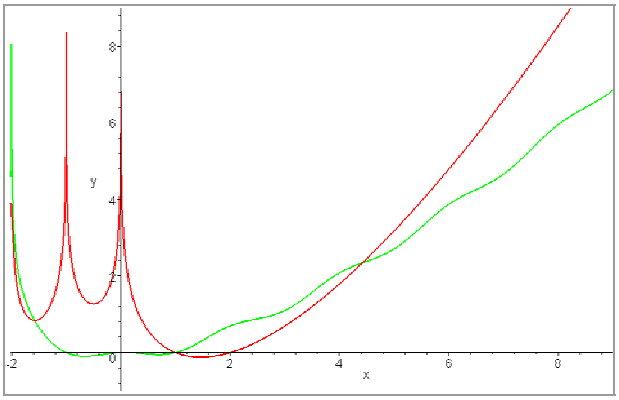
The product of all odd integers up to some odd positive integer n is often called the double factorial of n (even though it only involves about half the factors of the ordinary factorial, and its value is therefore closer to the square root of the factorial). It is denoted by n!!. For an odd positive integer n = 2k - 1, k ≥ 1, it is

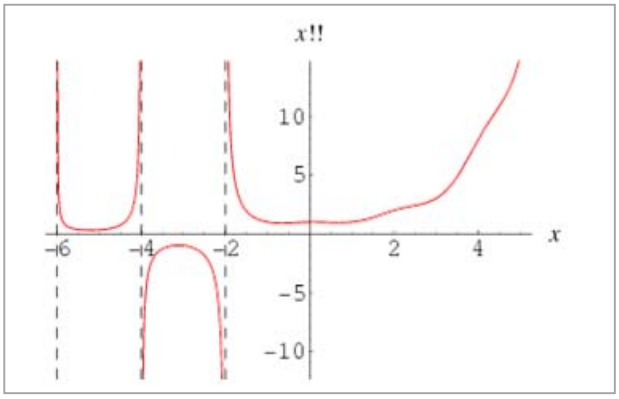
$$(2k-1)!! = \prod_{i=1}^{k}(2i-1)$$

A common related notation is to use multiple exclamation points to denote a multifactorial, the product of integers in steps of two (n!!), three (n!!!), or more. The double factorial is the most commonly used variant, but one can similarly define the triple factorial (n!!!) and so on. One can define the k-th factorial, denoted by $n!^{(k)}$, recursively for non-negative integers as:

$$n!^{(k)} = \begin{cases} 1, & \text{if } 0 \le n < k, \\ n((n-k)!^{(k)}), & \text{if } n \ge k\,, \end{cases}$$

The figures below show the plots for X!! (right), a comparison with log(abs(gamma)) (red) versus log(abs(doublegamma)) (green). – left.





Using the Logarithm is helpful to deal with large arguments, as these functions go beyond the calculator numeric range very quickly. Also ran out of space in the module to have more than one function on this subject, thus **LGMF** was chosen given its more general-purpose character.

The implementation is thru an all-MCODE function, yet execution times may be large depending on the arguments.

LGMF may also be used to compute factorials, use n=1 and then E^X on the result. Obviously the accuracy won't be the greatest, but it's a reasonable compromise

Stack	Input	Output
Y	n	/
X	x	LGMF(x)

Examples:

2 ENTER^ , 100 XEQ "**LGMF**" -> Log (100 !!) = 79.53457468
999 ENTER^, 123456 XEQ "**LGMF**" -> Log (123456 ! ..... !) = 578.0564932

And now lets move to something completely unrelated: Fourier Series representation of a given function, f(x).

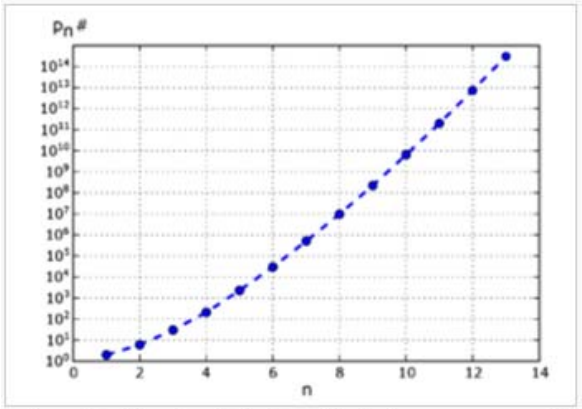
Appendix 5.- Primorials – a primordial view.

Welcome to the intersection between factorials and prime numbers...

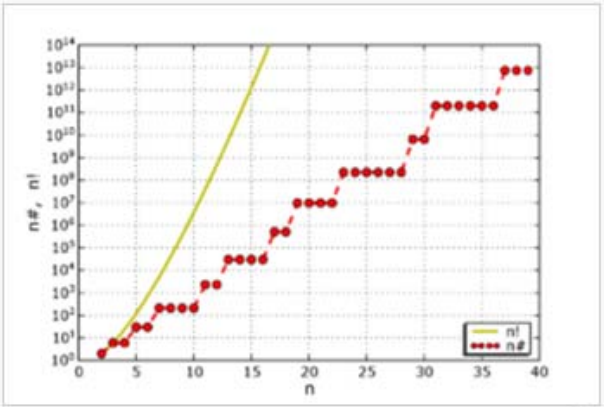
In number theory primorial is a function from natural numbers to natural numbers similar to the factorial function, but rather than multiplying successive positive integers, only successive prime numbers are multiplied. The name "primorial", attributed to Harvey Dubner, draws an analogy to primes the same way the name "factorial" relates to factors.

There are two conflicting definitions that differ in the interpretation of the argument: the first interprets the argument as an *index into the sequence of prime numbers* (so that the function is strictly increasing), while the second interprets the argument as *a bound on the prime numbers to be multiplied* (so that the function value at any composite number is the same as at its predecessor).

The figures below plot both definitions, comparing their shape and slopes:-



p_n# as a function of *n*, plotted logarithmically.



n# as a function of *n* (red dots), compared to *n*!. Both plots are logarithmic.

Prime primorial (left plot): For the nth prime number pn the primorial pn# is defined as the product of the first n primes (where pk is the kth prime number):

$$p_n\# = \prod_{k=1}^{n} p_k$$

Natural primorial (right plot): In general, for a positive integer n such a primorial n# can also be defined, namely as the product of those primes ≤ n.

$$n\# = \prod_{i=1}^{\pi(n)} p_i = p_{\pi(n)}\#$$

The FOCAL programs below can be used to calculate both flavors of primorials. Note the primordial (pun intended) role of function **PRIME?**, which effectively makes this a simple application as opposed to a full-fledge program from the scratch.

Examples: Calculate both primorials for the first 20 natural numbers.

Table of primorials

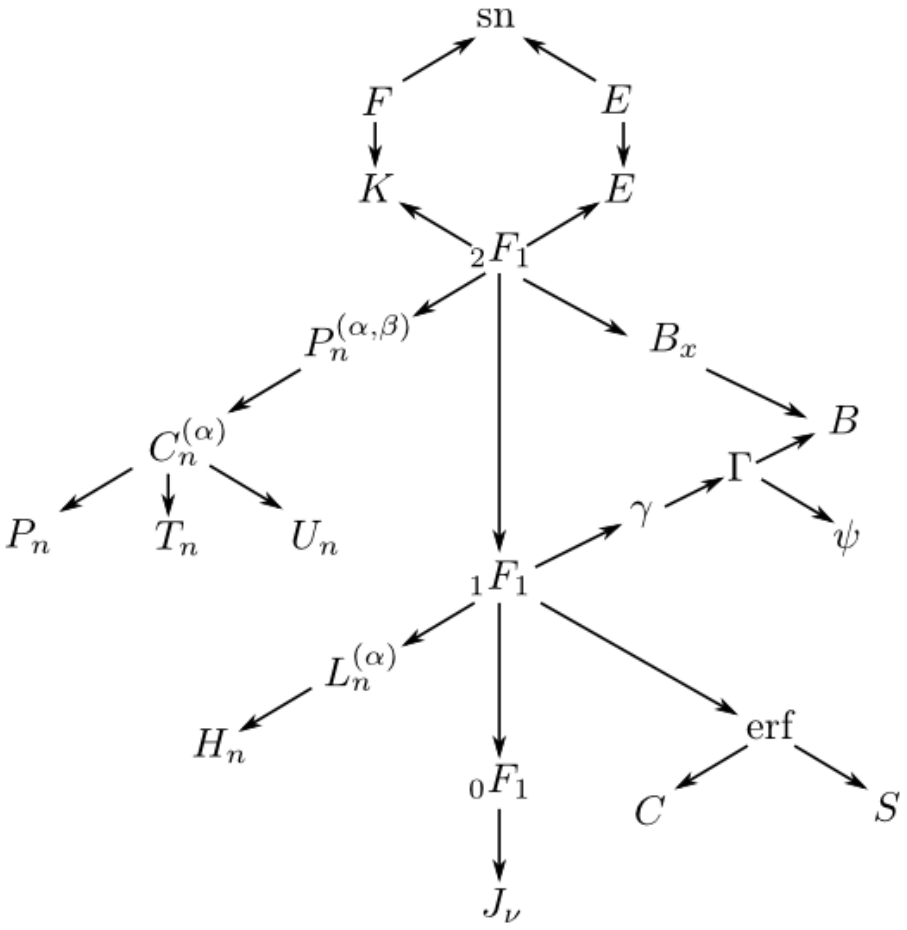
n	*n#*	p_n	p_n#
0	1	no prime	1
1	1	2	2
2	2	3	6
3	6	5	30
4	6	7	210
5	30	11	2310
6	30	13	30030
7	210	17	510510
8	210	19	9699690
9	210	23	223092870
10	210	29	6469693230
11	2310	31	200560490130
12	2310	37	7420738134810
13	30030	41	304250263527210
14	30030	43	13082761331670030
15	30030	47	614889782588491410
16	30030	53	32589158477190044730
17	510510	59	1922760350154212639070
18	510510	61	117288381359406970983270
19	9699690	67	7858321551080267055879090
20	9699690	71	557940830126698960967415390

01	LBL "NPRML"
02	ABS
03	INT
04	E
05	X>Y?
06	RTN
07	X<>Y
08	LBL 00
09	PRIME?
10	*GTO 01*
11	X<> L
12	*GTO 03*
13	LBL 01
14	ST* Y
15	LBL 03
16	DSE X
17	*GTO 00*
18	X<>Y
19	RTN

01	LBL "PPRML"
02	ABS
03	INT
04	E
05	X>Y?
06	RTN
07	STO Z
08	LBL 00
09	INCX
10	PRIME?
11	*GTO 01*
12	X<> L
13	*GTO 00*
14	LBL 01
15	ST* Z
16	DSE Y
17	*GTO 00*
18	RCL Z
19	END

Both routines only use the stack – no data registers or user flags are used. Clearly the numeric range will again be the weakest link, reaching it for n=54 for **PPRML** and n=251 for **NPRML**.

A glimpse of what's ahead:



"Relationship between common special functions". Taken from John Cook's web site:
http://www.johndcook.com/special_function_diagram.html

3.3. HIGH LEVEL MTH.

A word about the approach. The Hyper-Geometric Function as a generic function generator (or "*the case of the chameleon function in disguise*").

Special functions are particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications, frequently as solutions to differential equations. There is no general formal definition, but the list of mathematical functions contains functions which are commonly accepted as special. Some elementary functions are also considered as special functions.

The implementations described in this manual do nothing but scratching the surface (or more appropriately, "gingerly touching it") of the Special Functions field, where one can easily spend several life-times and still be just half-way through.

Implementing multiple special functions in a 41 ROM is clearly challenged by the available space in ROM, the internal accuracy and the speed of the CPU. It is therefore understandable that more commonality and re-usable components identified will make it more self-contained and powerful, overcoming some of the inherent design limitations.

The Generalized Hyper-geometric function is one of those rare instances that works in our favor, as many of the special functions can be expressed as minor variations of the general case. Indeed there are no less than 20 functions implemented as short FOCAL programs, really direct applications of the general case - saving tons of space and contributing to the general consistency and common approach in the SandMath.

We have Jean-Marc Baillard to thank for writing the original **HGF+**, the Generalized Hyper-geometric function - real cornerstone of the next two sections. The SandMath has an enhanced MCODE implementation that optimizes speed and accuracy thanks again to internal usage of 13-digit OS routines. The reuse made of it more than pays off for its lengthy code.

A few examples will illustrate this:-

$$\operatorname{erf}(x) = \frac{2x}{\sqrt{\pi}}\,{}_1F_1\left(\tfrac{1}{2}, \tfrac{3}{2}, -x^2\right).$$

$$J_\alpha(x) = \frac{(x/2)^\alpha}{\Gamma(\alpha+1)}\,{}_0F_1(\alpha+1; -\tfrac{1}{4}x^2).$$

$$\mathbf{H}_\alpha(z) = \frac{(z/2)^{\alpha+1/2}}{\sqrt{2\pi}\Gamma(\alpha+3/2)}\,{}_1F_2(1, 3/2, \alpha+3/2, -z^2/4)$$

Naturally this is not the case for any special function, and even it there's such an expression it may be more appropriate to use the direct definition instead – or an alternative one – for the implementation. This is the case of the Bessel functions, which use the series definition in the SandMath; the Gamma function using the Lanczos formula, etc.

With that said, let's delve into the individual functions comprising the High-Level Math group. First off come *those more frequently used so that they have gained their place in the ROM's main FAT.* Looking at the authorship you'll see the tight collaboration between JM and the author, as stated in the opening statements of this manual.

3.3.1. Gamma function and associates.

Let's further separate these by logical groups, depending on their similarities and applicability. The first one is the **GAMMA** and related functions: **1/GM**, **PSI**, **PSIN**, **LNGM**, **ICGM**, **BETA**, and **ICBT** – all of them a Quantum leap from the previous functions described in the manual, both in terms of the mathematical definition and as it refers to the required programming resources and techniques.

	Function	Author	Description
[ΣF]	**1/GMF**	*JM Baillard*	Reciprocal Gamma (Continuous fractions)
[ΣF]	**BETA**	*Ángel Martin*	Euler's Beta function
[ΣF]	**GAMMA**	*Ángel Martin*	Euler's Gamma function (Lanczos)
[ΣF]	**ICBT**	*JM Baillard*	Incomplete Beta function
[ΣF]	**ICGM**	*JM Baillard*	Incomplete Gamma function
[ΣF]	**LNGM**	*Ángel Martin*	Logarithm Gamma function
[ΣF]	**PSI**	*Ángel Martin*	Digamma (Psi) function
[ΣF]	**PSIN**	*JM Baillard*	Polygamma function

In mathematics, the Gamma function (represented by the capital Greek letter Γ) is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers.

If n is a positive integer, then

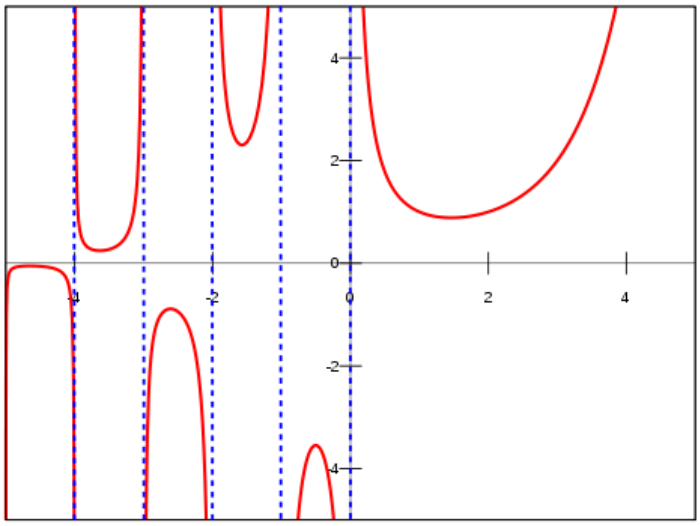
$$\Gamma(n) = (n-1)!$$

showing the connection to the factorial function.

For a complex number z with positive real part, the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$$

Things become much more interesting in the negative semi-plane, as can be seen in the plot on the right for real arguments.



The Gamma function has become standard in pocket calculators, either as extended factorials or as proper gamma definition. It's already available in the HP-11C and of course on the 15C, and that has continued to today's models. Implementing it isn't the issue, but achieving a reasonable accuracy is the challenge.

A popular method uses the Stirling approximation to compute Gamma. This is relatively simple to program, but its precision suffers for small values of the argument. A version suitable for calculators is as follows:

$$\Gamma(z) \approx \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e} \sqrt{z \sinh \frac{1}{z} + \frac{1}{810 z^6}} \right)^z$$

Valid for Re(z)>0, and with reasonable precision when Re(z)>8.

For smaller values than that it's possible to use the recurrence functional equation, taking it to the "safe" region and back-calculating the result with the appropriate adjusting factor:

$$\Gamma(z+1) = z\,\Gamma(z)$$

Incidentally, this method can be used for any approximation method, not only for Stirling's.

The method used on the SandMath is the Lanczos approximation, which lends itself better to its implementation and can be adjusted to have better precision with careful selection of the number of coefficients used. For complex numbers on the positive semi-plane [Re(z)>0], the formula used is as follows:

$$\Gamma(z) = \frac{\sum_{n=0..N} q_n z^n}{\prod_{n=0..N} (z+n)} (z+5.5)^{z+0.5} e^{-(z+5.5)}$$

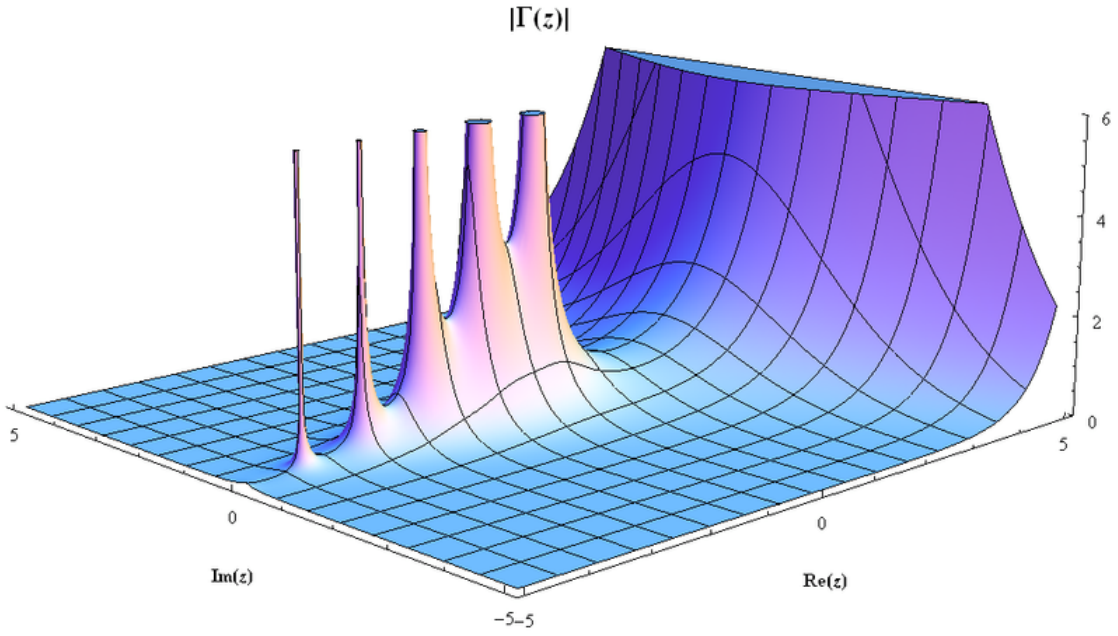
$q_0 =$	75122.6331530
$q_1 =$	80916.6278952
$q_2 =$	36308.2951477
$q_3 =$	8687.24529705
$q_4 =$	1168.92649479
$q_5 =$	83.8676043424
$q_6 =$	2.5066282

Although the formula as stated here is only valid for arguments in the right complex half-plane, it can be extended to the entire complex plane by the reflection formula,

$$\Gamma(1-z)\,\Gamma(z) = \frac{\pi}{\sin \pi z}.$$

An excellent reference source is found under http://www.rskey.org/gamma.htm, written by Viktor T. Toth.

Let's mention that this method yields good enough a precision that doesn't require using the functional equation to adjust it for small values of the argument. The obvious advantage is that without the required program loop, the execution time is shorter and constant for any input. This becomes of extreme importance when Gamma is used as a subroutine of more complex cases, like the Bessel J and I functions – where the cumulative additional time is very noticeable.



Appendix 6.- Accuracy comparison of different Gamma implementations.

The tables below provide a clear comparison between three methods used to calculate the Gamma function:

1. Lanczos formula, with k=6
2. Continuous fractions, and
3. Windschitl (Stirling).

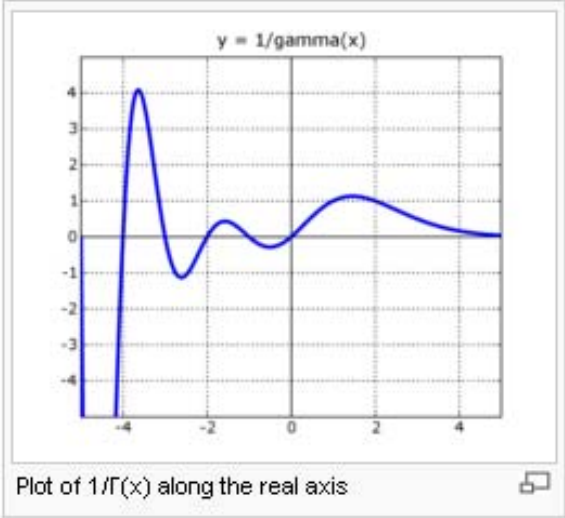
Each of them implemented using both standard 10-digit and enhanced 13-digit precision routines.

The results clearly show that the best implementation is Lanczos, and that the 13-digit routines provide a second order of magnitude improvement to the accuracy, or in other words: that it cannot compensate for the deficiencies of the used method. We're lucky in that the more accurate method is faster that the second best, albeit not as fast as Stirling's.

Obviously the extrapolation from integer case to the general case for the argument is assumed to follow the same trend, albeit not shown in the summary tables.

Standard 10-digit Implementation							
Reference		Lanczos (k=6)		Continuous Fractions		Windschitl (Stirling)	
(x-1) !	x	Result	error	Result	error	Result	err
1	1	1,000000001	1E-09	1,000000001	1E-09	1,000000012	1,2E-08
1	2	1	0	1,000000001	1E-09	1,000000012	1,2E-08
2	3	2	0	2,000000001	5E-10	2,000000024	1,2E-08
6	4	5,999999999	-1,66667E-10	6,000000002	3,33333E-10	6,000000071	1,18333E-08
24	5	24,00000001	4,16667E-10	24	0	24,00000028	1,16667E-08
120	6	120	0	120	0	120,0000014	1,16667E-08
720	7	720,0000008	1,11111E-09	720,0000001	1,38889E-10	720,0000087	1,20833E-08
5040	8	5040,000002	3,96825E-10	5040	0	5040,00006	1,19048E-08
40320	9	40320,00003	7,44048E-10	40319,99999	-2,4802E-10	40320,00048	1,19048E-08
362880	10	362880,0002	5,51146E-10	362879,9998	-5,5115E-10	362879,9988	-3,30688E-09
3628800	11	3628800,001	2,75573E-10	3628800,018	4,96032E-09	3628800,05	1,37787E-08
39916800	12	39916799,99	-2,50521E-10	39916800,01	2,50521E-10	39916800,9	2,25469E-08
479,001,600	13	479001599,5	-1,04384E-09	479001598,3	-3,549E-09	479001580,2	-4,1336E-08
6,227,020,800	14	6227020803	4,81771E-10	6227020798	-3,2118E-10	6227020957	2,52127E-08
Enhanced 13-digit Implementation							
Reference		Lanczos (k=6)		Continuous Fractions		Windschitl (Stirling)	
(x-1) !	x	Result	error	Result	error	Result	error
1	1	1	0	1	0	1	0
1	2	1	0	1,000000001	1E-09	1	0
2	3	2	0	2	0	1.999999999	-5E-10
6	4	6	0	6.000000004	6,66667E-10	5.999999997	-5E-10
24	5	24	0	24	0	23.99999999	-4,16667E-10
120	6	120	0	120	0	120,0000014	1,16667E-08
720	7	720	0	720	0	719.9999996	-5,55556E-10
5040	8	5040	0	5039,9999990	-1,9841E-10	5,039.999998	-3,96825E-10
40320	9	40320	0	40,320.00001	2,48016E-10	40,319.99998	-4,96032E-10
362880	10	362880	0	362880	0	362,880	0
3628800	11	3628800	0	3628800	0	3,628,800	0
39916800	12	39916800	0	39916800	0	39,916,799.99	-2,50521E-10
479,001,600	13	479001600	0	479001600	0	479,001,599.8	-4,17535E-10
6,227,020,800	14	6227020800	0	6227020800	0	6,227,020,800	0

3.3.2. Reciprocal Gamma function.



Plot of 1/Γ(x) along the real axis

The reciprocal Gamma function is the function

$$f(z) = \frac{1}{\Gamma(z)},$$

where Γ(z) denotes the Gamma function. Since the Gamma function is meromorphic and nonzero everywhere in the complex plane, its reciprocal is an entire function. The reciprocal is sometimes used as a starting point for numerical computation of the Gamma function, and a few software libraries provide it separately from the regular Gamma function.

Taylor series expansion around 0 gives

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + \left(\frac{\gamma^2}{2} - \frac{\pi^2}{12}\right) z^3 + \cdots$$

The SandMath however uses the expression based in continuous fractions, according to which:

```
Γ(x) = [x^(x-1/2)] sqrt(2π) exp [ -x + (1/12)/(x +
        + (1/30)/(x + (53/210)/(x +(195/371)/(x + ...  )))))]
```

Comparing the results obtained by GAMMA (using Lanczos) and continuous fractions it appears that ***the precision is generally better in the Lanczos case*** – which also happens to be faster due to its polynomial-like form and the absence of loops to adjust the result for smaller arguments.

Note the special case for x=0, which is not a pole for this function but it is a singularity for all the others that used the common subroutines – therefore the dedicated check in the routine listing.

3.3.3. (Lower) Incomplete Gamma function.

In mathematics, the upper and the lower incomplete gamma functions are respectively as follow:

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\, e^{-t}\, \mathrm{d}t. \qquad \gamma(s,x) = \int_0^x t^{s-1}\, e^{-t}\, \mathrm{d}t.$$

Connection with Kummer's confluent hypergeometric function, when the real part of z is positive,

$$\gamma(s,z) = \; s^{-1} z^s e^{-z} M(1, s+1, z)$$

which is the expression used in the SandMath.

The Upper incomplete Gamma function can be easily obtained from the relationship:

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s).$$

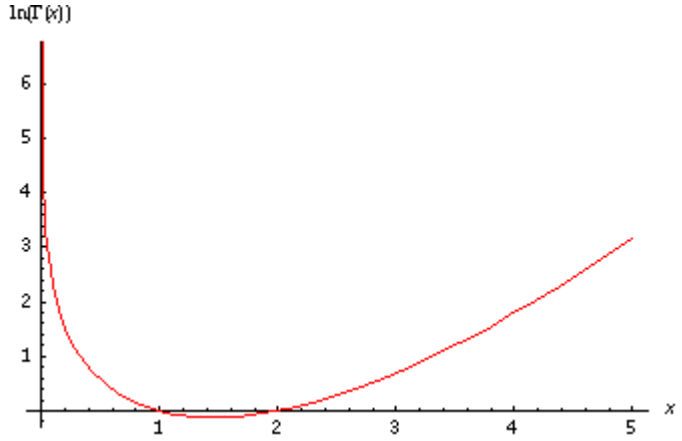
3.3.4. Log Gamma function.

Many times is easier to calculate the Logarithm of the Gamma function instead of the main Gamma value. This could be due to numeric range problems (remember that the 41 won't support numbers over E100), or due to the poles and singularities of the main definition.

The SandMath uses the Stirling approximation to compute LogGamma, as given by the following expression (directly obtained from the formula in page 27):

$$2\ln\Gamma(z) \approx \ln(2\pi) - \ln z + z\left(2\ln z + \ln\left(z\sinh\frac{1}{z} + \frac{1}{810z^6}\right) - 2\right)$$

This approximation is also good to more than 8 decimal digits for z with a real part greater than 8. For smaller values we'll use the functional equation to extend it to the region where it's accurate enough and then back-calculate the result as appropriate.



The picture on the left shows the LogGamma function for positive arguments. Interestingly it has a negative results region between 1 and 2 – so it isn't always positive.

Note also the asymptotic behavior near the origin – due to the Gamma function pole.

The implementation on the SandMath uses the analytical continuation to calculate LogGamma for arguments less than 9, *including negative values*. Obvious problems (like the poles at negative integer) will yield DATA ERROR messages, but outside that the approximation should hold.

since: $\Gamma(z+n) = \Gamma(z) * \Sigma(z+i) \mid_{i=1,2..n}$

it follows: $\text{Ln}\,\Gamma(z+n) = \text{Ln}\,\Gamma(z) + \text{Ln}\,[\Sigma(z+i) \mid_{i=1,2..n}]$

Notice also that the same error will occur when trying to calculate LogGamma when Gamma is negative, which occurs between even-negative numbers and their immediately lower (inferior) one – see the plot in page 27).

See the following link for a detailed description of another implementation (using Lanczos for both cases) to calculate Gamma and LogGamma on the 41 by Steven Thomas Smith:
http://www.hpmuseum.org/cgi-sys/cgiwrap/hpmuseum/articles.cgi?read=941

An excellent implementation of Gamma and related functions for the 41 is available on the following link, written by Jean-Marc Baillard (very complete and detailed):
http://www.hpmuseum.org/software/41/41gamdgm.htm

3.3.5. Digamma function.

In mathematics, the digamma function is defined as the logarithmic derivative of the gamma function:

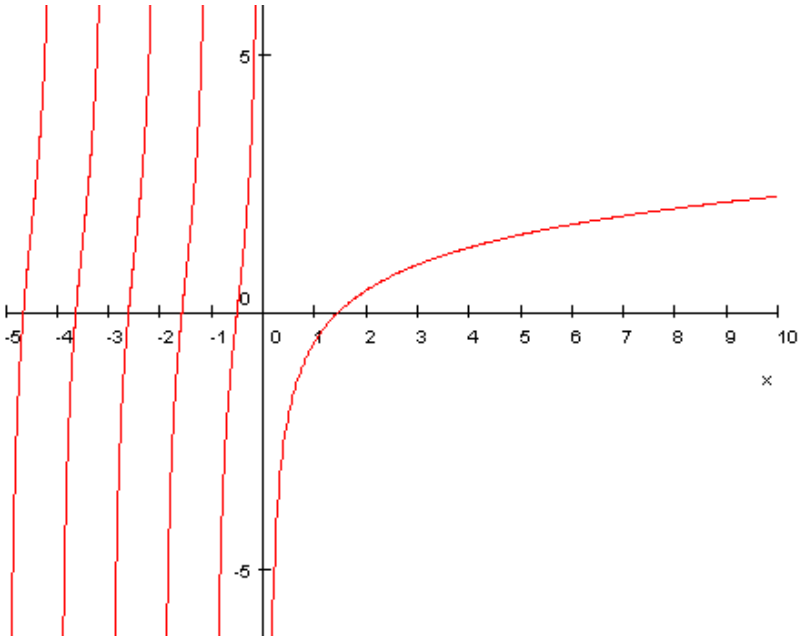
$$\Psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

It is the first of the polygamma functions.

Its relationship to the harmonic numbers is shown in that for natural numbers:

$$\Psi(n) = H_{n-1} - \gamma$$

where Hn is the n'th harmonic number, and γ is the Euler-Mascheroni constant.



As can be seen in the figure on the left plotting the digamma function, it's an interesting behavior showing the same poles and other singularities to worry about. It should be possible to find an approximation valid for all the definition range of the function.

It has been implemented on the SandMath using the formulas derived from the called Gauss digamma theorem, although further simplified in the following algorithm:

$$\Psi(x) = \log(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + O\left(\frac{1}{x^8}\right)$$

programmed as: **u^2{[(u^2/20-1/21)u^2 + 1/10]u^2 –1}/12 – [Ln u + u/2]**,

The implementation also makes use of the analytic continuation to take it to arguments greater than 9 (same as it's done for LogGamma), using the following recurrence relation to relate it to smaller values:

$$\Psi(x+1) = \Psi(x) + \frac{1}{x}.$$

Which naturally can be applied for negative arguments as well.

3.3.6. Euler's Beta function.

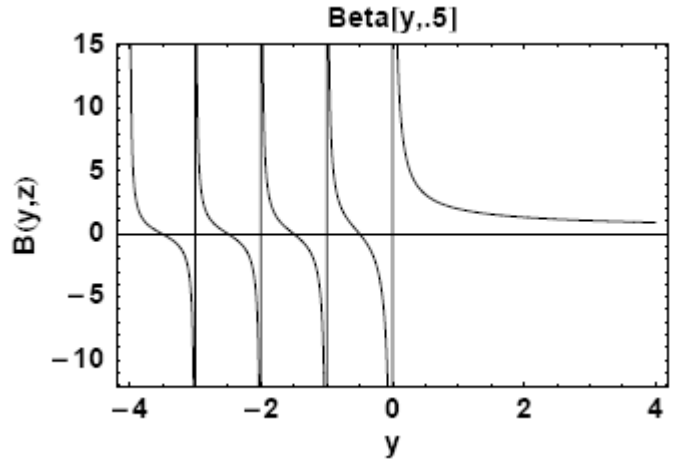
The beta function, also called the Euler integral of the first kind, is a special function defined by

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad \mathrm{Re}(x), \mathrm{Re}(y) > 0.$$

The beta function was studied by Euler and Legendre and was given its name by Jacques Binet. The most common way to formulate it refers to its relation to the Gamma function, as follows:

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

As a graphical example, the picture below shows B(X,0.5) for values of x between –4 and 4. As it's expected, the same Gamma problem points are inherited by the Beta function.



The implementation on the SandMath makes no attempt to discover new approaches or utilize any numeric equivalence: it simple applies the definition formula using the Gamma subroutine. Obvious disadvantages include the reduced numeric range – aggravated by the multiplication of gamma values in the numerator.

Execution time corresponds to three times that of the Gamma function, plus the small overhead to perform the Alpha Data checks and the arithmetic operations between the three gamma values.

3.3.7. Incomplete Beta Function.

The incomplete beta function, a generalization of the beta function, is defined as:

$$\mathrm{B}(x;\,a,b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt.$$

For x = 1, the incomplete beta function coincides with the complete beta function. The relationship between the two functions is like that between the gamma function and its generalization the incomplete gamma function.

And it's given in terms of the Hypergeometric function the expression by:

$$B\,(z;\,a,\,b) \;=\; \frac{z^a}{a}\;{}_2F_1\,(a,\,1-b;\,a+1;\,z)$$

3.3.8. Bessel functions and Modified.

The next logical group would be the Bessel functions – with special guests Riemann's ZETA and Lambert's W (both branches).

	Function	Author	Description
[ΣF]	**IBS**	*Ángel Martin*	Bessel I(n,x) of the first kind
[ΣF]	**JBS**	*Ángel Martin*	Bessel J(n,x) of the first kind
[ΣF]	**KBS**	*Ángel Martin*	Bessel K(n,x) of the second kind
	SIBS	*Ángel Martin*	Spherical Bessel i(n,x)
[ΣF]	**SJBS**	*Ángel Martin*	Spherical Bessel j(n,x)
[ΣF]	**SYBS**	*Ángel Martin*	Spherical Bessel y(n,x)
[ΣF]	**WL0**	*Ángel Martin*	Lambert's W - main branch
	WL1	*Ángel Martin*	Lambert's W – secondary branch
[ΣF]	**YBS**	*Ángel Martin*	Bessel Y(n,x) of the second kind
[ΣF]	**ZETA**	*Ángel Martin*	Riemann's Zeta – direct method
	ZETAX	*JM Baillard*	Riemann's Zeta – Borwein algorithm

The SandMath Module includes a set of functions written with the harmonic analysis in mind, specifically to facilitate the calculation of the Bessel functions in their more general sense: for any real number for order and argument.

Bessel functions of the First kind – I(n,x) and J(n,x)

The formulae used are as follows:

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

Where Γ denotes the Gamma function.

These expressions are valid for any real number as order, although there are issues for negative integers due to the singularities in the poles of the gamma function - as there's always a term for which (m+n+1) equals zero or negative integers, all of them being problematic.

To avoid this, we use the following expression for negative integer orders:

$$J_{-n}(x) = (-1)^n J_n(x).$$

Whilst: $I_{-\alpha}(x) = I_\alpha(x)$, for every real number order.

This definition is also valid for negative values for X, as there's no singularity for any x value.

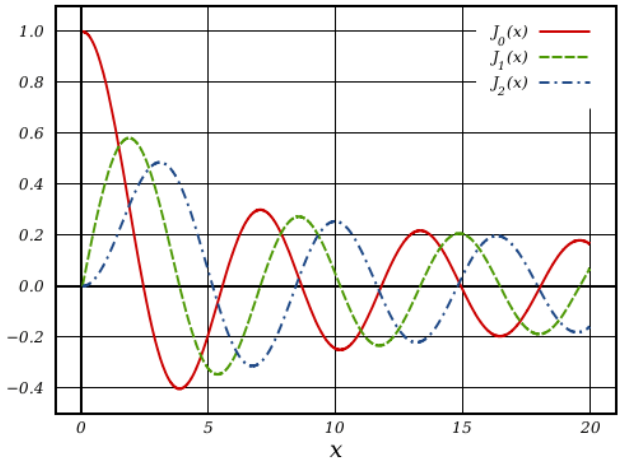
The SandMath implementation uses a recurrence formula instead of the one shown above. It has the clear advantage of not having to calculate Gamma for each term in the sum, contributing to a much faster and robust algorithm.

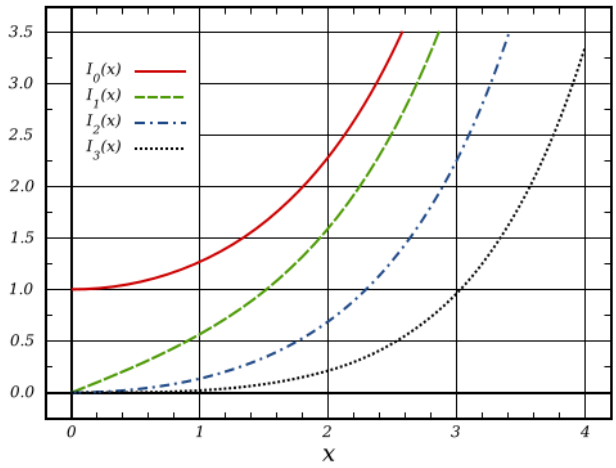
The iterative relationships are as follows:

$J(n,x) = \Sigma\{U(k)|k=1,2\ldots\} * (x/2)^n / \Gamma(n+1)$, where:

$U(k) = - U(k-1) * (x/2)^2 / k(k+n)$, with **$U(0) = 1$.**

The graphics below plot the Bessel functions of the first kind, $J_\alpha(x)$, and their modified, $I_\alpha(x)$, for integer orders $\alpha=0,1,2,...$





Note that *for large values of the argument, the order or both* these algorithms will return **incorrect results for J(n,x)**. This is due to the alternating character of the series, which fools the convergence criteria at premature times and fouls the intermediate results. Unfortunately there isn't an absolute criteria for validity, but a practical rule of thumb is to doubt the result if (n+x) it greater than 20.

Bessel functions of the Second kind – K(n,x) and Y(n,x)

The formulae used are as follows:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}. \qquad K_\alpha(x) = \frac{\pi}{2}\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}$$

These expressions are valid for any real number as order – with the same issues as the first kind functions above when the order is integer. To avoid the singularities and to reduce the calculation time, *the following expressions are used for integer orders*:

$$\pi\, \mathbf{Y_n}(x) = 2[\gamma + \mathbf{Ln}\ x/2]\ \mathbf{J_n}(x) - \Sigma\{(-1)^k\ \mathbf{f_k}(n,x)\} - \Sigma\{\mathbf{g_k}(n,x)\}$$

$$2\ \mathbf{K_n}(x) = (-1)^{n+1}\ 2\ [\gamma + \mathbf{Ln}\ x/2]\ \mathbf{I_n}(x) + (-1)^n\ \Sigma\{\ \mathbf{f_k}(n,x)\} + \ \Sigma\{(-1)^k\ \mathbf{g_k}(n,x)\}$$

where γ is the Euler–Mascheroni constant (0.5772...), and:

$$\mathbf{g_k}(n,x) = (x/2)^{2k-n}\ \ \mathbf{(n-k-1)!\ /\ k!}\ \ ;\ \boldsymbol{k=0,1,2,\ldots(n-1)}$$

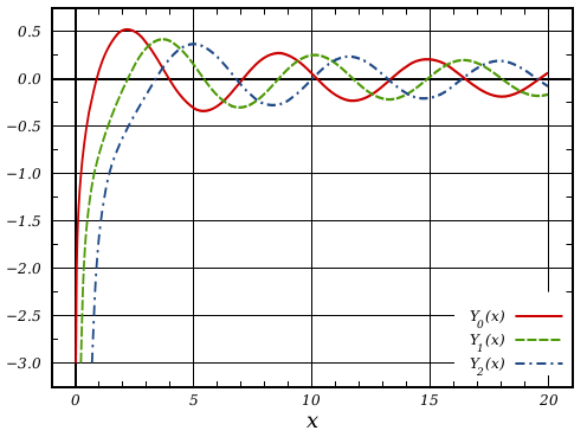
$$\mathbf{f_k}(n,x) = (x/2)^{2k+n}\ [\mathbf{H}(k) + \mathbf{H}(n+k)]\ /\ [\mathbf{k!}\ (n+k)!]\ \ ;\ \boldsymbol{k=0,1,2,\ldots..}$$

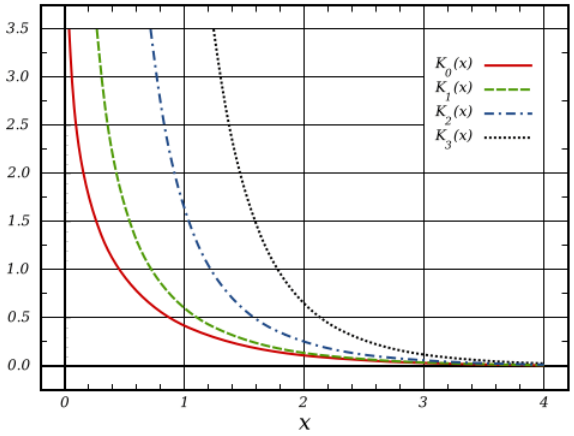
and H(n) is the *harmonic number*, defined as: $\mathbf{H(n)} = \Sigma\mathbf{(1/k)}\ |\ k = 1,2,..\ n$

Where: $Y_{-n}(x) = (-1)^n Y_{\boldsymbol{n}}\ \boldsymbol{(x)},$ and $K_{-n}\ (x) = K_n(x)$

(*) note that for x<0, Y(n,x) and K(n,x) are complex numbers.

The graphics below plot the Bessel functions of the second kind, $Y_\alpha(x)$, and their modified, $K_\alpha(x)$, for integer orders α=0,1,2,...





Note that **KNBS** and **YNBS** are FOCAL programs that use dedicated MCODE functions specially written for the calculations (**#BS** and **#BS2**). Their entries are located in the sub-functions FAT, thus won't be shown in the main CAT listings – in case you wonder about their whereabouts.

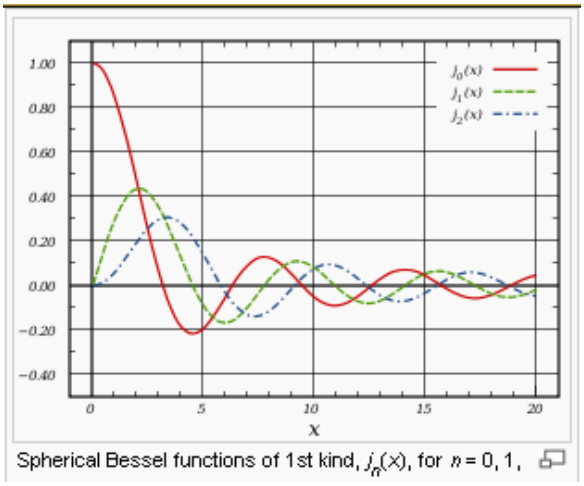
Getting Spherical, are we?

The spherical Bessel functions jn and yn, and are (very closely) related to the ordinary Bessel functions Jn and Yn by:

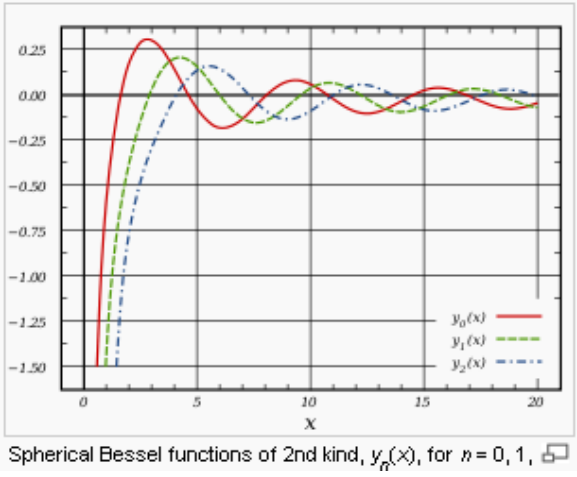
$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x),$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x) = (-1)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x).$$

Which graphical representation (naturally very JBS-ish looking) is show below:



Spherical Bessel functions of 1st kind, $j_n(x)$, for $n = 0, 1,$



Spherical Bessel functions of 2nd kind, $y_n(x)$, for $n = 0, 1,$

Notice that there really isn't any Spherical i(n,x) properly defined – but there's one in the SandMath just the same, using the same relationship as for j(n.x) and y(n,x).

Once again, remember than as (n+x) increases the accuracy of the results decreases – specially for J(n,x), Y(n,x) and the spherical counterparts, where the returned value can be completely incorrect if (n+x) > 20 (a practical rule, not an absolute criterion).

Programming Remarks.

The basic algorithms use the summation definition of the functions, calculating the successive values of the sum until there's convergence for the maximum precision (10 decimal places on the display). Therefore the execution time can take a little long – a fact that becomes a non-issue on the CL. Or when using 41-emulator programs, like V41, setting the turbo mode on.

There are different algorithms depending on whether the order is integer or not. This speeds up the calculations and avoids running into singularities (as mentioned before).

Note that for integer indexes the gamma function changes to a factorial calculation, which benefits from faster execution on the calculator. Non-integer orders utilize the special MCODE function, **GAMMA**, with shorter execution times than equivalent FOCAL programs – but still longer than **FACT** when integers.

Besides that, for integer orders the execution time is further reduced by ***calculating simultaneously the two infinite sums involved*** in the first kind and the second kind terms. This assumes that the convergence occurs at comparable number of terms, which fortunately is the case - given *their relative fast convergence*.

Note that in order to obtain similar expressions for both **Yn** and **Kn** – and so getting simpler program code - we can re-write Kn as follows:

$$(-1)^{n+1}\ 2\ K_n(x) = 2\ [\gamma + \mathbf{Ln}\ x/2]\ I_n(x)\ - \Sigma\{\ f_k(n,x)\} - \ (-1)^n\ \Sigma\{(-1)^k\ g_k(n,x)\}$$

Dedicated MCODE Functions.

To further decrease the execution time of the programs, two dedicated functions have been written, implemented as MCODE routines as follows*:*

Function	Flag 00 Clear	Flag 00 Set
#BS	$\Sigma\ U_k(n,x)$, $\|k=0,1,2$	where $U_k = -\ U_{k-1} * (x/2)^2 / k(k+n)$
#BS2	$\Sigma\{f_k(n,x)\}$ $\|k=0,1,2...$	or: $\Sigma\{g_k(n,x)\}$ $\|k=0,1,...(n-1)$

The first function **#BS** is used equally in the calculation of the first kind and the second kind of non-integer orders.

Function	Integer	Non-integer
JBS **IBS**	**#BS**	
YBS **KBS**	2x **#BS2**	**#BS**

As it was said before, the summation will continue until the contribution of the newer term is negligible to the total sum value. All calculations are done using the full 13-digit precision of the calculator. No rounding is made until the final comparison, which is done on 10-digit values.

From the definition above it's clear that **#BS** coincides with either Jn(x) or In(x) depending on the status of the CPU flag 9, and for positive orders. The functions **JBS** and **IBS** are just MCODE extensions of **#BS** that set up the specific settings prior to invoking it, and (depending on the signs of the orders and the arguments) possibly adjust the result after it's completed.

The second function **#BS2** is only used for second kind functions with integer orders. It's a finite sum, and not an infinite summation. Its contribution to the final result grows as the function order increases. Its main goal was to reduce execution time as much as possible, derived from the speed gains of MCODE versus FOCAL.

The definition of $f_k(n,x)$ is as follows:

$$\mathbf{f}_k(n,x) = \{(x/2)^{2k+n} \;/\; [k!\,(n+k)!]\} \;[\mathbf{H}(k) + \mathbf{H}(n+k)] \;;\; k=0,1,2\ldots$$

The definition of gk(n,x) is as follows:

$$\mathbf{g}_k(n,x) = (x/2)^{2k-n} \;(n-k-1)! \;/\; k! \;\;;\; k=0,1,\ldots(n-1)$$

The calculation of Gamma uses the Lanczos approximation implemented in the **GAMMA** function of the SandMath. Despite being reasonably fast, its execution time is noticeably longer than that of the Factorial for integer indexes – therefore **#BS2** will use **FACT** instead for integer orders.

The Harmonic Numbers **H(n)** are obtained using another SandMath function as subroutine, **Σ1/N**. You see that the internals of **#BS2** perform quite an involved procedure, utilizing multiple resources within the SandMath module.

Furthermore, **#BS2** is called **twice** within the FOCAL program to calculate **KBS** or **YBS** – once for the first, infinite summation and a second time for the second, finite sum. The status of User Flag 00 controls the calculation made. That was done to save one FAT entry, when the limiting factor was the maximumm nuber of functions per page (i.e. 64 functions). Now they have been pushed eveh further off, to the secondary FAT used for the sub-functions group.

Bessel Function	Summed Functions by #BS2	Flag 00	Flag 01
Yn(x)	$g_k(n,x)$	Set	Set
	$f_k(n,x)$	Clear	
Kn(x)	$(-1)^k * g_k(n,x)$	Set	Clear
	$(-1)^k * f_k(n,x)$	Clear	

Note also that for this case (integer orders) there's two infinite summations involved for the Bessel functions of the second kind. This is done simultaneously within **#BS2** *when user flag 02 is set,* as both series converge in very similar conditions (i.e. with the same number of terms).

Main functions: IBS, JBS, KBS, and YBS.

The first kind pair (IBS and JBS) are 100% written in MCODE – including exception handling and special cases. This is the only version known to the author of a full-MCODE implementation on the 41 platform, and it is however a good example of the capabilities of this machine.

No data registers are used for **IBS** and **JBS** – but both the stack and the Alpha registers are used. The number of terms required for the convergence is stored in register N upon termination.

The second kind pair (**KBS** and **YBS**) is implemented using a FOCAL driver program for **#BS** and **#BS2**. Notably more demanding than the previous two, their expressions require additional calculations that exceed the reasonable MCODE capabilities.

Although they're not normally supposed to be used outside of the Bessel program, **#BS** and **#BS2** could be called independently. Both use the same input parameters: index in Y and half of the argument in X. Pay close attention to the status of user flags 00 and 01 as they directly influence their result.

Other functions used in the Bessel calculations, which aren't part of the native HP-41 function set, are as follows:

GEU	Euler's constant = 0,577215665
CHSYX	CHS repeated n times (in X register) multiplied by Y register.
INT? and **FRC?**	Conditional based on value in X being integer or fractional
E3/E+	Self-explanatory: divides X by 1,000, then adds one.

Examples:

J(1,1) = 0,440050586
J(-1,-1) = 0,440050586
J(0.5,0.5) = 0,540973790
J(-0.5, 0.5) = 0,990245881

I(1,1) = 0,565159104
I(-1,-1) = -0,565159104
I(0.5,0.5) = 0,587993086
I(-0.5,0.5) = 1,272389647

Y(1,1) = -0,781212821
Y(-1,2) = 0,107032431
Y(0.5,0.5) = -0,990245881
Y(-0.5,0.5) = 0,540973790

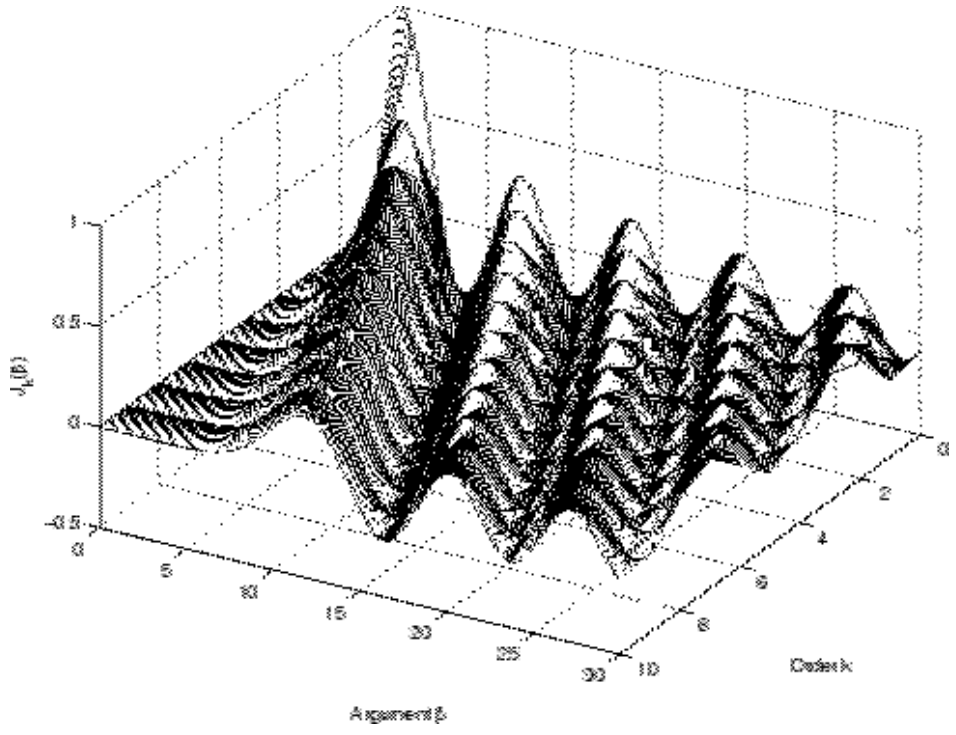
K(1,1) = 0,601907230
K(-1,2) = 0,139865882
K(0.5,0.5) = 1,075047604
K(-0.5,0.5) = 1,075047604

Error Messages:

Note that the functions will return a "*DATA ERROR*" message when the solution is a complex number, like J(-0.5, -0.5) or I(-0,5, -0.5). There's no way around that save in some particular cases of the order. You can always use the versions available in the 41Z Module for a full complex range.

"*OUT OF RANGE*", when the calculator numeric range is exceeded. This typically occurs for large indexes, during the power exponentiation step.

"*ALPHA DATA*" indicates alphabetic data in registers X or Y. May also trigger "*DATA ERROR*".



Appendix 7.- FOCAL program used to calculate the Bessel Functions of the second kind. As you can see it's just a simple driver of the MCODE functions, with the additional task of orchestrating the logic for the different cases.

Note the usage of the sub-functions from the auxiliary FAT , as well as other SandMath functions.

```
01  LBL "KBS"
02  SF  01
03  GTO 00
04  LBL "YBS"
05  CF  01
06  LBL 00
07  X=0?
08  RTN             single case x=0
09  2
10  /               HALFX
11  STO  03         x/2
12  X<>Y            swap things
13  STO 01          n
14  INT?            is it integer order?
15  GTO 02          yes, divert to section
16  CHS             -n
17  X<>Y            x/2
18  RAD
19  SPFC#           Multi-Function Launcher
20  1               Recurrence Sum #BS
21  CHS             -J(-n,x)
22  STO 02          partial result
23  RCL 01          n
24  RCL 03          x/2
25  SPFC#           Multi-Function Launcher
26  1               Recurrence Sum #BS
27  STO 00          save J(n,x) here - used by Hankel
28  FC?  01         is KBS?
29  GTO 01          yes, skip
30  RCL 01          n
31  PI
32  *
33  COS
34  *
35  LBL 01
36  RCL 02          partial result
37  +
38  RCL 01          n
39  PI
40  *
41  SIN
42  /
43  FS?  01         is YBS?
44  GTO 03
45  2
46  /               HALFX
47  PI
48  *
49  CHS
50  GTO 03

51  LBL 02          orden,argument swapped
52  CF  02          default case
53  X<0?            is it negative?
54  SF  02          negative order
55  ABS             remember this fact!
56  STO 01          abs(n)
57  ,
58  STO 00          reset counter
59  STO 02          and partial sum
60  RDN
61  X=0?            skip if n=0
62  GTO 06
63  CF  00          selects  #B2
64  SPFC#           Σ[gk(n,x)] |k=0,1...(n-1)
65  2               #BS2
66  CHS
67  STO 02
68  RCL 01          abs(n)
69  LBL 06
70  RCL 03          x/2
71  SF  00          selects #B1
72  SPFC#           Σ[fk(n,x)] |k=0,1,2...
73  2               #BS2
74  ST- 02          partial result
75  RCL 03
76  LN
77  GEU
78  +
79  RCL*            showing off ! :-)
80  ST+  X(3)
81  ST+ 02          partial result
82  RCL 02
83  FS?  01         is it YBS?
84  GTO 04          yes, cut the chase
85  RCL 01          abs(n)
86  E
87  +               INCX
88  CHSYX           (-1)^n+1 * result
89  2
90  /               HALFX
91  GTO 03
92  LBL 04
93  PI
94  /
95  FC?  02         was negative order?
96  GTO 03          no, skip correction
97  RCL 01          abs(n)
98  CHSYX
99  LBL 03
100 STO 02          final result
101 END
```

3.3.9. Riemann Zeta function.

Perhaps one of the most-studied functions in mathematics, it owes its popularity to its deep-rooted connections with prime numbers theory. Not having an easy approximation to work with, its implementation on the 41 will be a bit of a challenge – mainly due to the very slow convergence of the series representation used to program it. Be assured that this numeric calculation won't help you prove the Riemann hypothesis (and collect the $1M prize) – so adjust your expectations accordingly.

The Riemann zeta function is a function of complex argument **s** that analytically continues the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

or the integral form:

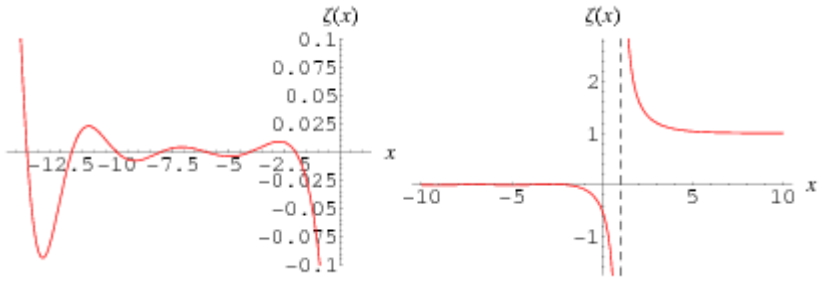
$$\zeta(x) \equiv \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{u^{x-1}}{e^u - 1}\, du,$$

The Riemann zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s),$$

valid for all complex numbers s (excluding 0 and 1), which relates its values at points s and 1−s.

The plots below of the real Zeta function show the negative side with some trivial zeros, as well as the pole at x=1.



The direct implementation in the SandMath module uses the alternative definitions shown below, in a feeble attempt to get a faster convergence (which in theory it does although not very noticeably given the long execution times involved). The summations are called the Dirichlet Lambda and Eta functions respectively.

$$(1 - 2^{-x})\,\zeta(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x} \qquad \zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

Go ahead and try **ZETA** with FIX 9 set in the calculator – you'll see the successive iterations being shown for each additional term, until the final result doesn't change. Be aware than MCODE or not, *it'll take a very long time for small arguments,* approaching infinite as x approaches zero.

For values lower than 1 we make use of the following relationship – a sort of "reflection formula" if you wish.

The interesting fact about this is how it has been implemented: if x<1 then the MCODE function branches to a FOCAL program that (as part of the calculations) calls the MCODE function after doing the change: x = (1-x), which obviously is >1.

$$\zeta(x) = \begin{cases} \sum_{k=1}^{\infty} k^{-x}, & \text{per } x > 1 \\ 2^x \pi^{x-1} \sin\left(\frac{x\pi}{2}\right) \Gamma(1-x)\,\zeta(1-x), & \text{per } x < 1 \end{cases}$$

Really the direct method isn't very useful at all, and it's more of an anecdotal implementation with academic value but not practical. The Borwein algorithm provides an iterative alternative to the direct method, with a much faster convergence even as a FOCAL program, and more comfortable treatment. It is implemented in the SandMath as a courtesy of JM Baillard, in the function **ZETAX**.

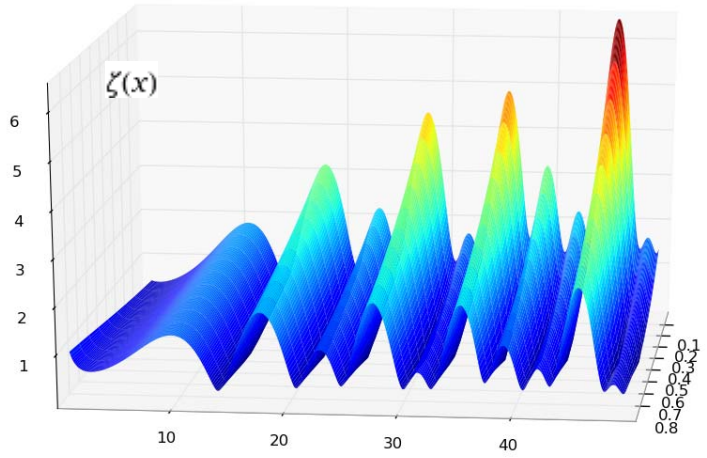
For example, using **ZETAX** to calculate Z(1.001) returns the correct solution is a few seconds!
See the appendices for a FOCAL listing of the program if interested.

Examples.-

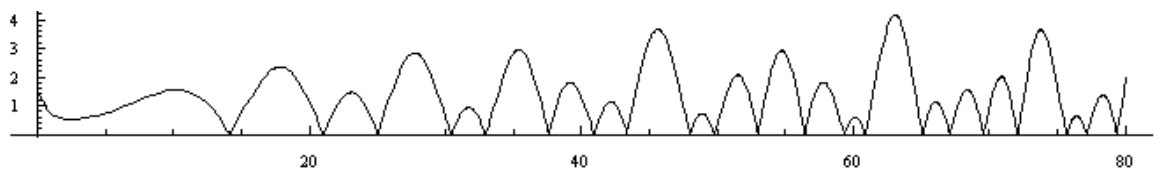
Complete the table below for ζ(x), using both the direct method and the Borwein algorithm. Use the result in WolframAlfa as reference to also determine their respective errors.

x	ζ(x)	Direct	error	Borwein	error
-5	-0,0039682539682	-0,003968254	8,0136E-09	-0,003968254	8,0136E-09
5	1,036927755	1,03692775	-4,96019E-09	1,036927755	-1,38255E-10
3	1,202056903	1,20205676	-1,19096E-07	1,202056903	-1,32764E-10
2	1,6449340668482	n/a	n/a	1,644934066	-5,15644E-10
1,1	10,58444846	n/a	n/a	10,58444847	4,77115E-10

We see that not only is the Borwein algorithm faster and more capable in range, but also their results are more accurate than the direct approach; MCODE or not, 13-digit internal subroutines notwithstanding.



Note: The following links to the MAA and the (now defunct) Zetagrid make fascinating reading on the Zeta zeros current trends and historic perspective – make sure you don't miss them!
http://www.maa.org/editorial/mathgames/mathgames_10_18_04.html
http://www.zetagrid.net/



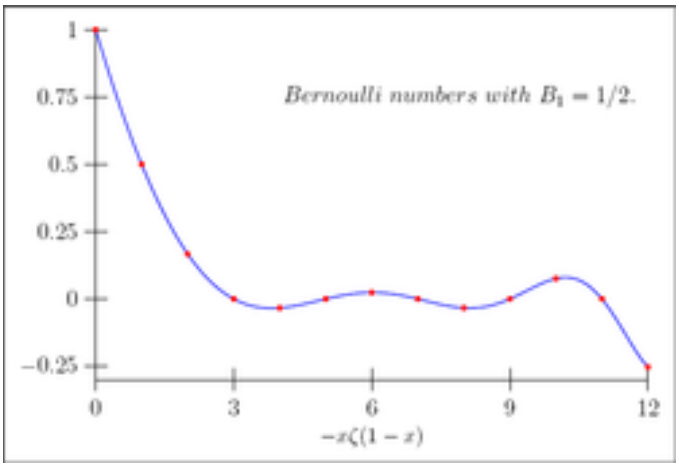
Appendix 8.- Putting Zeta to work: Bernoulli numbers.

In mathematics, the Bernoulli numbers Bn are a sequence of rational numbers with deep connections to number theory. The values of the first few Bernoulli numbers are

B0 = 1, B1 = ±1/2, B2 = 1/6, B3 = 0, B4 = −1/30, B5 = 0, B6 = 1/42, B7 = 0, B8 = −1/30.

If the convention B1=−1/2 is used, this sequence is also known as the first Bernoulli numbers; with the convention B1=+1/2 is known as the second Bernoulli numbers. Except for this one difference, the first and second Bernoulli numbers agree. Since Bn=0 for all odd n>1, and many formulas only involve even-index Bernoulli numbers, some authors write Bn instead of B2n.

The Bernoulli numbers were discovered around the same time by the Swiss mathematician Jakob Bernoulli, after whom they are named, and independently by Japanese mathematician Seki Kōwa. Seki's discovery was posthumously published in 1712 in his work Katsuyo Sampo; Bernoulli's, also posthumously, in his Ars Conjectandi of 1713. Ada Lovelace's note G on the analytical engine from 1842 describes an algorithm for generating Bernoulli numbers with Babbage's machine. As a result, the Bernoulli numbers have the distinction of being the subject of the first computer program.



There are several (or rather many!) algorithms and approaches to the calculation of Bn. In this particular example we'll use the expression based on the Riemann's Zeta function, according to which the values of the Riemann zeta function satisfy

$$n\,\zeta(1-n) = -Bn$$

for all integers n≥0. The expression n ζ(1 − n) for n = 0 is to be understood as the limit of x ζ(1 − x).) when x->0.

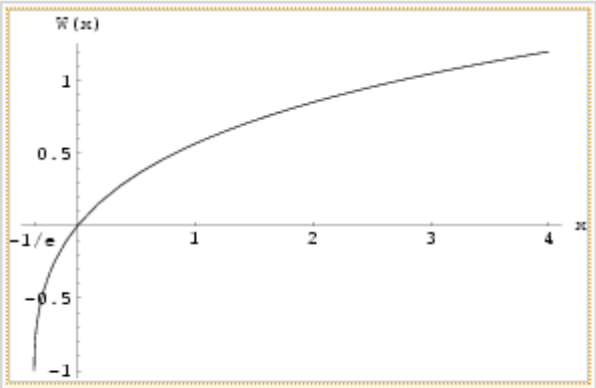
The FOCAL program on the right is a super-short application of the **ZETA** function, even if it's used for negative arguments, Obviously we've single-cased the troublesome points to avoid execution times unreasonably long, but apart from that it's quite generic in its approach. It also uses a few others SandMath functions as additional bonus.

```
01  LBL "BN2"
02  X=1?
03  GTO 01
04  X=0?
05  INCX
06  X=1?
07  RTN
08  ODD?
09  CLX
10  X=0?
11  RTN
12  2
13  X<Y?
14  GTO 00
15  6
16  1/X
17  RTN
18  LBL 00
19  ST+ X
20  X<Y?
21  GTO 00
22  -30
23  1/X
24  RTN
25  LBL 00
26  X<>Y
27  LBL 01
28  STO M
29  CHS
30  INCX
31  ZETA
32  RCL M
33  CHS
34  *
35  END
```

3.3.10. Lambert W function.

The last function deals with the implementation of the Lambert W function. Oddly enough its definition is typically given as the inverse of another function, as opposed to having a direct expression. This makes it a bit backwards-looking initially but in fact it is significantly easier to implement than the Riemann Zeta seen before.

The Lambert W function, named after Johann Heinrich Lambert, also called the Omega function or product log, is the inverse function of f(w) = w exp(w) where exp(w) is the natural exponential function and w is any complex number. The function is denoted here by W.



The graph of $W_0(x)$ for $-1/e \le x \le 4$

For every complex number z:

$$z = W(z)e^{W(z)}.$$

The Lambert W function cannot be expressed in terms of elementary functions. It is useful in combinatory, for instance in the enumeration of trees.

It can be used to solve various equations involving exponentials and also occurs in the solution of delay differential equations.

The Taylor series of W0 around 0 can be found using the Lagrange inversion theorem and is given by:

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}\, x^n$$

where n! is the factorial. However, this series oscillates between ever larger positive and negative values for real z>~0.4, and so cannot be used for practical numerical computation.

The W function may be approximated using Newton's method, with successive approximations to w = W(z) (so z = w e^w) being:

$$w_{j+1} = w_j - \frac{w_j e^{w_j} - z}{e^{w_j} + w_j e^{w_j}}.$$

The implementation in the SandMath uses this iterative method to solve for W(z) the roots of its functional equation, given the functions argument z. An important consideration is the selection of the initial estimations. For that the general practice is to start with Ln(x) as lower limit, and 1+Ln(x) as upper value.

Another aspect of the W function is the existence of two branches. The second branch is defined for arguments between –1/e and 0, with function values between –1 and –infinite.

The "lower" branch is also available in the SandMath as the function **WL1**. In fact the MCODE algorithm is the same one, with just different initial estimations depending on the branch to calculate!.

Example 1: calculate W for x=5

5, **WL0** -> "RUNNING…", followed by 1,326724665

We can use the inverse Lambert function **AWL** to check the accuracy of the results, simply executing it after WL0 and comparing with the original argument. Note the **AWL** will be seen later on, in the Secondary FAT (Sub-functions) group. This it requires **ΣFL$** to call it, not **XEQ**.

5, **WL0**, **ΣFL$** "**AWL**" -> 4,999999998; an error of err= 4 E-10

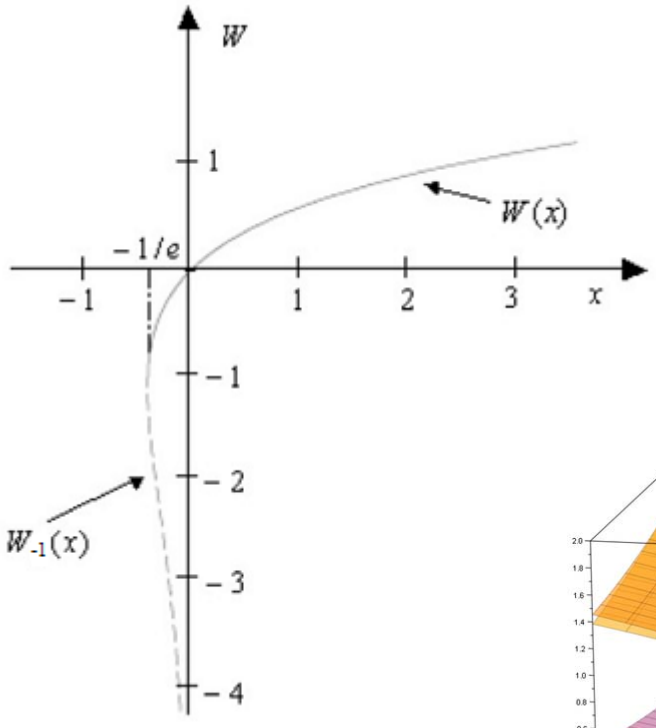
where **ΣFL$** can be called using the main launcher: **[ΣFL], [**ALPHA**]**

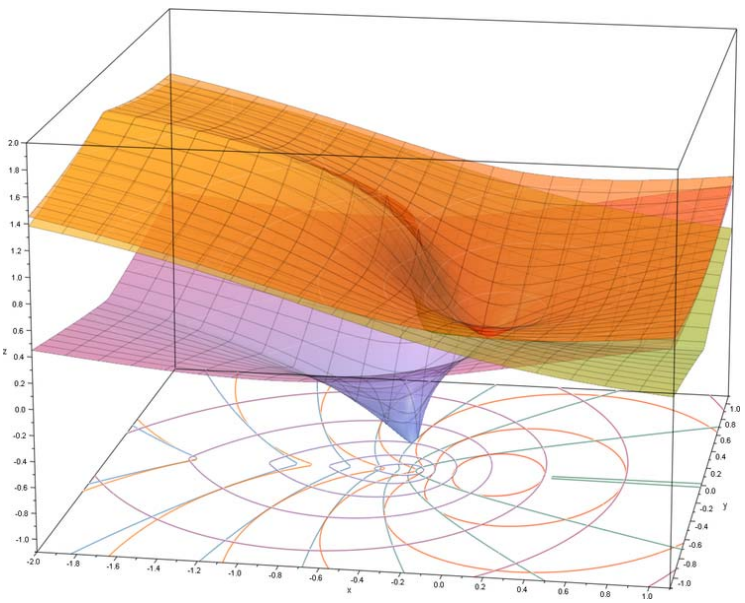
Example 2.- calculate the Omega constant, ω = W(1)

1, **WL0** => "RUNNING…" , followed by 0,567143290

Example 3: Calculate both branches of W for x=-1/2e

1, E^X, CHS, ST+ X, 1/X, **WL0** -> W_0 (-1/2e) = -0,231960953
LASTX, XEQ "**WL1**" -> W_{-1}(-1/2e) = -2,678346990





And here's a 3D representation of the complex Lambert to end this section with a graphical splash. Enough to make you want to start using your 41Z Module, isn't it?

3.4. Remaining Special Functions in the Main FAT.

The third and last chapter of the Special functions in the main FAT comprises other Hyper-geometric derived functions, plus a couple of notable exceptions not easy to associate: **FFOURN** and **LINX**

	Function	Author	Description
[ΣF]	**CI**	*JM Baillard*	Cosine Integral
[ΣF]	**EI**	*JM Baillard*	Exponential Integral
[RF]	**ELIPF**	*Ángel Martin*	Eliptic Integral
[ΣF]	**ERF**	*JM Baillard*	Error Function
	FFOUR	*Ángel Martin*	Fourier coefficients for (x)
[H]	**HCI**	*JM Baillard*	Hyperbolic Cosine Integral
	HGF+	*JM Baillard*	Generalized Hyper-geometric Function
[H]	**HSI**	*JM Baillard*	Hyperbolic Sine Integral
	LINX	*Ángel Martin*	Dilogarithm function
[ΣF]	**SI**	*JM Baillard*	Sine Integral

Notable examples of "multi-purposed function" are also the Carlson Integrals, themselves a generator for several other functions like the Elliptic Integrals. More about this in the corresponding paragraphs later on.

Exponential Integral and associates.

The first sub-section covers the Exponential, Logarithmic, Trigonometric and Hyperbolic integrals. They're all calculated using their expressions using the Generalized Hyper-geometric function, in a clear demonstration of the usefulness or the adopted approach.

For real nonzero values of x, the exponential integral Ei(x) is defined as:

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt.$$

Integrating the Taylor series for exp(t) and extracting the logarithmic singularity, we can derive the following series representation for real values:

$$\mathrm{Ei}(x) = \gamma + \ln|x| + \sum_{k=1}^{\infty} \frac{x^k}{k\ k!} \qquad x \neq 0$$

where we substitute the series by its Hyper-Geometric representation:

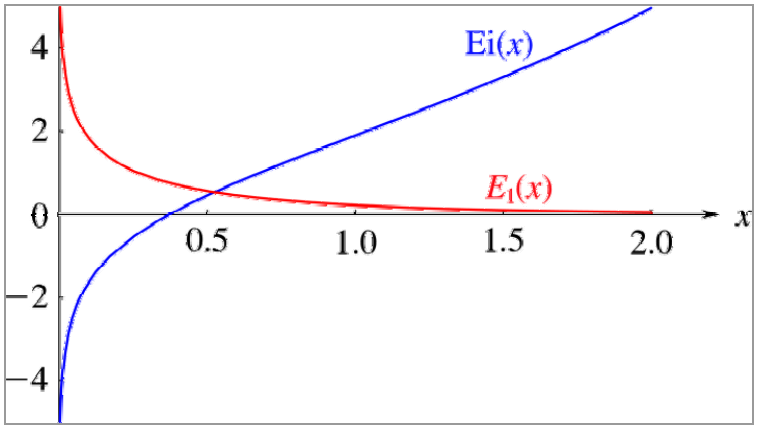
$$\Sigma\{\ x\text{^}k\ /\ k\ k!\} = \ x\ *\ {}_2F_2(\ 1,\ 1\ ;\ 2,\ 2;\ x\ )$$

The logarithmic integral has an integral representation defined for all positive real numbers by the definite integral:

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}.$$

The function li(x) is related to the exponential integral Ei(x) via the equation:

$\mathrm{li}(x) = \mathrm{Ei}(\ln x)$, which is the one used to program it in the SandMath module.



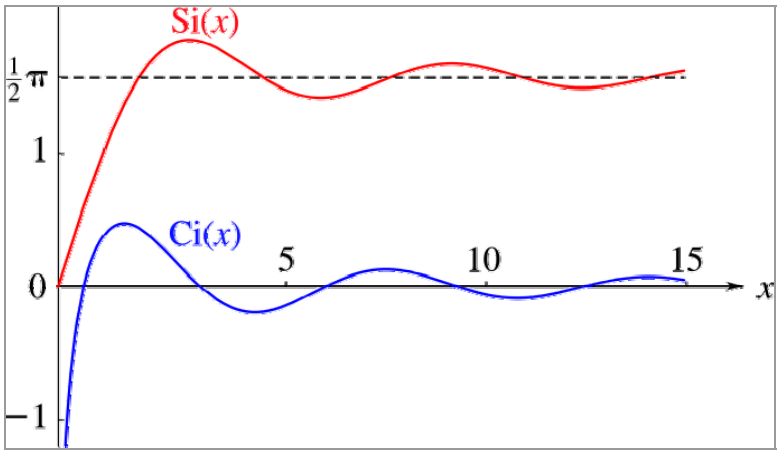
The different trigonometric and hyperbolic integral definitions and their relations with the Hyper-Geometric funcion (for the relevant integral in the definition) are as follows:

$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$	$\mathrm{Shi}(x) = \int_0^x \frac{\sinh t}{t}\, dt$
$x * {}_1F_2(1/2;\ 3/2,\ 3/2;\ -x^2/4)$	$x * {}_1F_2(1/2\ ;\ 3/2\ ,\ 3/2\ ;\ x^2/4)$

$\mathrm{Ci}(x) = \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t}\, dt$	$\mathrm{Chi}(x) = \gamma + \ln x + \int_0^x \frac{\cosh t - 1}{t}\, dt$
$-(x^2/4)\ {}_2F_3(1,\ 1\ ;\ 2,\ 2,\ 3/2\ ;\ -x^2/4)$	$(x^2/4)\ {}_2F_3(1,\ 1\ ;\ 2,\ 2,\ 3/2\ ;\ x^2/4)$

Examples:

1.4 XEQ "SI" -> Si(1.4) = 1.256226733 - or: [ΣF], [**Z**]
1.4 XEQ "CI" -> Ci(1.4) = 0.462006585 - or: [ΣF], [**V**]
1.4 XEQ "SHI" -> Shi(1.4) = 1.561713390 - or: [ΣF], [SHIFT], [**Z**]
1.4 XEQ "CHI" -> Chi(1.4) = 1.445494076 - or: [ΣF], [SHIFT], [**V**]



Even if it's not covered by the SandMath, the following relation between the Exponential and Trigonometric Integrals is available:

$$\mathrm{E}_1(ix) = i\left(-\tfrac{1}{2}\pi + \mathrm{Si}(x)\right) - \mathrm{Ci}(x) \qquad (x > 0)$$

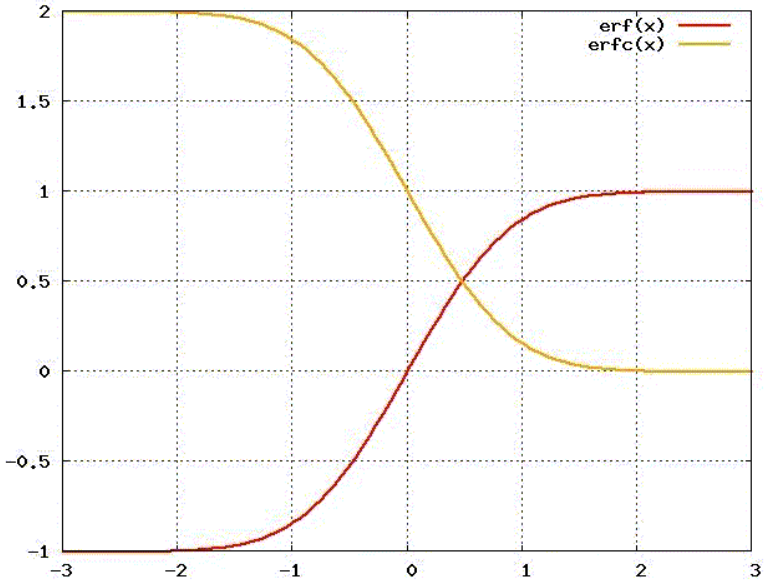
Errare humanum est.

In mathematics, the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape which occurs in probability, statistics and partial differential equations. Its definition and the expression based on the Hyper-geometric function (via ascending series) are given in the table below:

$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$	erf x = $(2x/\pi^{1/2})\ \exp(-x^2)\ \ {}_1F_1(\ 1,\ 3/2\ ;\ x^2\ )$

The complementary error function, denoted erfc, is defined as : $\operatorname{erfc} = 1 - \operatorname{erf}(x)$

Both functions are shown below for an overview.



The unsung hero: HGF+

If we're to believe that behind a great man there is often an even greater woman, then the greatest idea behind all these functions is the implementation of the Generalized Hyper-geometric function. A general-purpose definition requires the use of data registers for the parameters (a1... am) and (b1, ... bn) , and expects the argument x in the X register, and the number of parameters m and n is stored in Z and Y, for the generic expression:

$$ {}_mF_p(a_1,a_2,....,a_m\ ;\ b_1,b_2,....,b_p\ ;\ x\ ) = $$
$$ = \ \Sigma_{k=0,1,2,.....}\ [(a_1)_k(a_2)_k.....(a_m)_k] \ /\ [(b_1)_k(b_2)_k.....(b_p)_k]\ .\ x^k/k! $$

- If m = p = 0 , **HGF+** returns exp(x)
- The program doesn't check if the series are convergent or not.
- Even when they are convergent, execution time may be prohibitive: press any key to stop
- Stack register T is saved and x is saved in L-register.
- R00 is unused.
- The alpha "register" is cleared.

The original **HGF+** was written by Jean-Marc Baillard. Only small changes have been made to the version in the SandMath, optimizing the code and checking for ALPHA DATA in all registers used, as well as for the argument x.

Appendix 9.- Inverse Error Function.- coefficients galore...

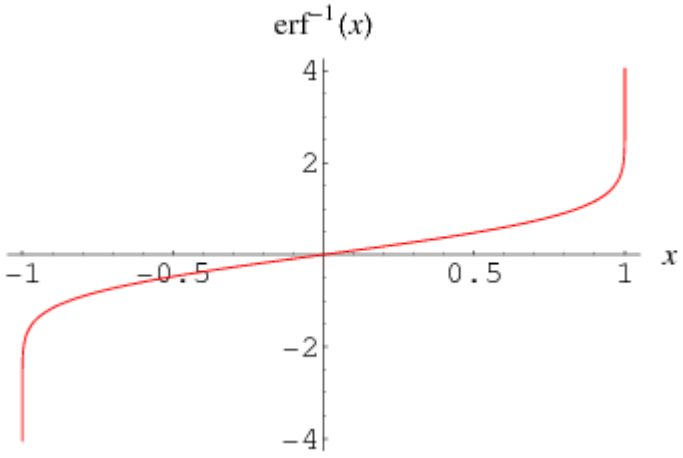
The inverse error function can be defined in terms of the Maclaurin series

$$\text{erf}^{-1}(z) = \sum_{k=0}^{\infty} \frac{c_k}{2k+1} \left(\frac{\sqrt{\pi}}{2} z \right)^{2k+1},$$

Where c0 = 1 and

$$c_k = \sum_{m=0}^{k-1} \frac{c_m c_{k-1-m}}{(m+1)(2m+1)} = \left\{ 1, 1, \frac{7}{6}, \frac{127}{90}, \ldots \right\}.$$

This really is a bear to handle, requiring quite a number of coefficients to be calculated for good accuracy result. Moreover, that calculation involves a lot of registers to store the values – since there isn't any iterative approach based on recursion.



The expression below is definitely too inaccurate (only three or four digits are correct) to deserve a dedicated MCODE function:

$$\text{erf}^{-1}(z) = \tfrac{1}{2}\sqrt{\pi} \left(z + \frac{\pi}{12} z^3 + \frac{7\pi^2}{480} z^5 + \frac{127\pi^3}{40320} z^7 + \frac{4369\pi^4}{5806080} z^9 + \frac{34807\pi^5}{182476800} z^{11} + \cdots \right).$$

A paper from 1968 by A. Strecok lists the first 200 coefficients of a power series that represents the inverse error function. While using this approach it became clear that at least 30 of them are needed for a 10-digit accuracy for 0< x < 0,85. This only gets worse as x approaches 1, getting into a clear example of the "law of diminishing results".

A better method for the vicinity of 1 is probably to use an asymptotic expansion, such as:

$$\text{erf}^{-1}(z) \propto \frac{1}{\sqrt{2}} \sqrt{\log\left(\frac{2}{\pi (z-1)^2}\right) - \log\left(\log\left(\frac{2}{\pi (z-1)^2}\right)\right)} \; /; (z \to 1)$$

A combination of both approaches would seem to be the best compromise, depending on the argument. . Typing the 30 coefficients is not fun however, thus the best is no doubt to use a data file in X-Memory to keep them safe.

How many logarithms, say again?

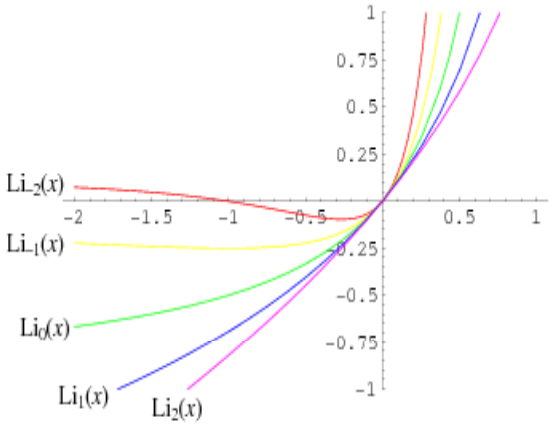
LINX calculates the polylogarithm function, (also known as Jonquière's function) a special function defined by the infinite sum, or power series:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots .$$

Only for special values of the order s does the polylogarithm reduce to an elementary function such as the logarithm function. The above definition is valid for all complex orders s and for all complex arguments z with |z| < 1; it can be extended to |z| ≥ 1 by the process of analytic continuation.

For particular cases, the polylogarithm may be expressed in terms of other functions (see below). Particular values for the polylogarithm may thus also be found as particular values of these other functions. For integer values of the polylogarithm order, the following explicit expressions are known:

$$\mathrm{Li}_1(z) = -\ln(1-z)$$
$$\mathrm{Li}_0(z) = \frac{z}{1-z}$$
$$\mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}$$
$$\mathrm{Li}_{-2}(z) = \frac{z\,(1+z)}{(1-z)^3}$$
$$\mathrm{Li}_{-3}(z) = \frac{z\,(1+4z+z^2)}{(1-z)^4}$$
$$\mathrm{Li}_{-4}(z) = \frac{z\,(1+z)(1+10z+z^2)}{(1-z)^5} .$$



The SandMath implementation is a direct series summation, adding terms until their contribution to the sum is negligible. Convergence is very slow, especially for small arguments. Its usage expects n to be in register Y and x in register X. The result is saved in X, and X is moved to LastX.

The program below gives a FOCAL equivalent – note the clever programming done by JM Baillard to only do one Y^X to reduce the execution times significantly.

```
01  LBL "LIN"        13  STO 03
02  STO 01           14  ISG 00
03  X<>Y             15  CLX
04  STO 02           16  RCL 00
05  1                17  RCL 02
06  STO 03           18  Y^X
07  CLX              19  /
08  STO 00           20  +
09  LBL 01           21  X#Y?
10  RCL 01           22  GTO 01
11  RCL 03           23 END
12  *
```

Fourier Series.

In mathematics, a Fourier series decomposes periodic functions or periodic signals into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials). The study of Fourier series is a branch of Fourier analysis.

The partial sums for f are trigonometric polynomials. One expects that the functions ΣN f approximate the function f, and that the approximation improves as N tends to infinity. The infinite sum

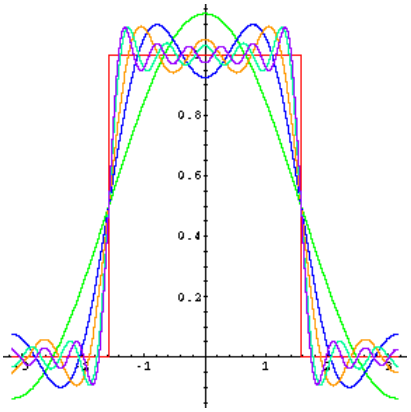
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

is called the Fourier series of f. The Fourier series does not always converge, and even when it does converge for a specific value x0 of x, the sum of the series at x0 may differ from the value f(x0) of the function. It is one of the main questions in harmonic analysis to decide when Fourier series converge, and when the sum is equal to the original function.

FFOUR Calculates the Fourier coefficients for a periodic function F(x), defined as:

$$a_n = \frac{1}{L} \int_0^{2L} f(x') \cos\left(\frac{n\pi x'}{L}\right) dx'$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x') \sin\left(\frac{n\pi x'}{L}\right) dx'.$$



with the following characteristics:

- centered in x = x0
- with period 2L on an interval [x0, x0+2L]
- with a given precision for calculations (significant decimal places)

FFOUR is a rather large FOCAL program, despite having a MCODE FAT entry. It calculates all integrals internally, not making use of general-purpose numeric integrators like INTEG, IG, etc – so it's totally self-contained.

The function must be programmed in main memory under its own global label. The program prompts for the first index to calculate, and the number of desired coefficients.

The program also calculates the approximate value of the function at a given argument applying the summation of the terms, using the obtained coefficients:

$$f(x') = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x'}{L}\right).$$

To use it simply enter the value of x and press "A" (XEQ A) in user mode on – this assumes that no function is assigned to the key. The approximation will be more correct when a sufficient number of terms is included. The goodness is also dependent on the argument itself.

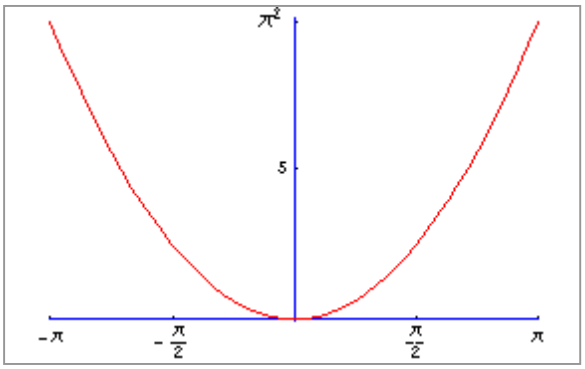
Example: calculate the first six coefficients for F(x) = x^ 2, assuming:

a period T=2pi, centered in x0 = 0. As it's known,

X^2 = 4/3 π^2 + **Σ{** 4 cos(nx) /n^2 - 4π sin(nx) /n **}** |*n=0,1,...*

Using an accuracy of 6 decimal places the program returns the following results:

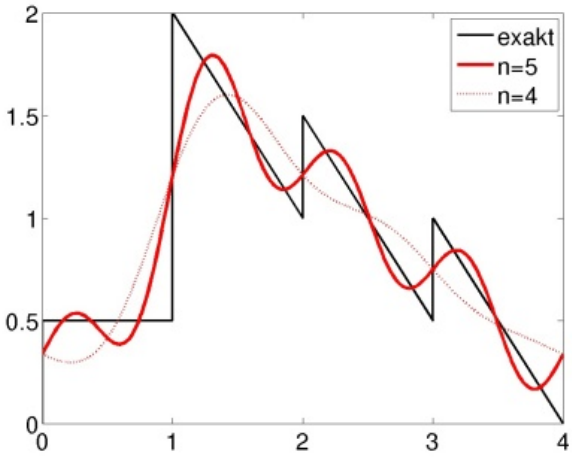
a0 = 13,1595	*b0 = 0*
a1 = 4	b1 = -12,566
a2 = 1	b2 = -6,5797
a3 = 0,4444	b3 = -4,1888
a4 = 0,250	b4 = -3,1415
a5= 0,160	b5 = -2.513



Pressing [A] will calculate an estimation of the function for the argument in X, using the fourier temrs calculated previously. In this case:

X=5, XEQ [A] -> f(x) = 23,254423
X=1, XEQ [A] -> f(x) = -0,154639, which obviously misses the point.

Typically the functions used are related to the harmonic analysis though. Here's an nteresting one, the "Christmas-Tree" function and its Fourier representation for different number of terms.



Appendix 10.- Fourier Coefficients by brute force.

Since the coefficients are basically integrals of the functions combined with trigonometric functions, nothing (besides common sense) stops us from using INTEG to calculate them. This brute force approach is just a work-around, considering the time requirements for the execution – but it can be useful to calcuate a single term randomly, as opposed to the sequential approach used by FFOUR.

So here the idea is to calculate the n-th. Coefficient independently, which responds to the following definig equation:

1	LBL "FOURN"
2	"F. NAME? "
3	AON
4	PROMPT
5	AOFF
6	ASTO 01
7	"PERIOD=?"
8	PROMPT
9	STO 00
10	FIX 4
11	LBL E
12	"INDEX=?"
13	PROMPT
14	STO 02
15	CF 00
16	XEQ 00
17	SF 00
18	LBL 00
19	"*FN"
20	0
21	RCL 00
22	INTEG
23	RCL 00
24	/
25	ST+ X
26	"a"
27	FS? 00
28	"b"
29	"|-("
30	RCL 02
31	AINT
32	"|-)="
33	ARCL Y
34	PROMPT
35	FC? 00
36	RTN
37	GTO E

38	LBL "*FN"
39	RAD
40	STO 03
41	XEQ IND 01
42	RCL 02
43	RCL 03
44	*
45	RCL 00
46	/
47	ST+ X
48	PI
49	*
50	FC? 00
51	COS
52	FS? 00
53	SIN
54	*
55	DEG
56	END

Example: f(x) = x^2

1	LBL "X2"
2	X^2
3	END

Example: f(x) = x

1	LBL "X"
2	END

Notice that the module SIROM ('Solve and Integrate" ROM) contains not only **FROOT** and **FINTG**, but also the program **FOURN** in its "-APPLIED" section – so you can use that 4k rom instead of the Advantage – that'll also save you from having to type in the program.

Simply enter the information asked at the prompts, including the precision desired (number of decimal digits), function name and its chosen period (2π).

The screenshot below shows the ILPER output of the process:

Printer

```
XROM "FOURN"
F.NAME?
X2 RUN
T=?
PI
 3,141593 ***
 2,000000 *
 6,283185 ***
RUN
PREC.=?
 6,000000 RUN
N=?
 7,000000 RUN
```

Using this program we'll calculate the coeffcients for the 7th and 9th terms for f(x) = x^2.

a7 = 0.081633, b7 = -1,795196; and:
a9 = 0,049383, b9 = -1,396263

3.5.- More Special Functions in the Secondary FAT

We've finally come to the last part of the manual, covering those functions included in the Sub-functions group with entries in the secondary (hidden) FAT. Go ahead and review the accessibility information from the introduction for a quick refresher if needed.

Let's divide and conquer – using the Carlson and Hankel launchers as grouping criteria.-

3.5.1. Carlson Integrals and associates.

The first sub-function launcher is the Carlson group. It's loosely centered on the Carlson's integrals, plus related functions. The launcher prompt is activated by pressing [**O**] at the main **ΣFL** prompt, and offers the following seven choices:

The table below shows in the first column the letter used for each of the functions within this group:

[CR]	Function	Author	Description
[E]	**ELIPF**	*Ángel Martin*	Elliptic Integral
[F]	**CRF**	*JM Baillard*	Carlson Integral 1st. Kind
[G]	**CRG**	*JM Baillard*	Carlson Integral 2nd. Kind
[J]	**CRJ**	*JM Baillard*	Carlson Integral 3rd. Kind
[C]	**CSX**	*JM Baillard*	Fresnel Integrals, C(x) & S(x)
[W]	**WEBAN**	*JM Baillard*	Weber and Anger functions
[Y]	**AIRY**	*JM Baillard*	Airy Functions Ai(x) & Bi(x)

The Elliptic Integrals.

In integral calculus, elliptic integrals originally arose in connection with the problem of giving the arc length of an ellipse. They were first studied by Giulio Fagnano and Leonhard Euler. Modern mathematics defines an "elliptic integral" as any function f which can be expressed in the form

$$f(x) = \int_c^x R\left(t, \sqrt{P(t)}\right)\, dt,$$

where R is a rational function of its two arguments, P is a polynomial of degree 3 or 4 with no repeated roots, and c is a constant.

The most common ones are the incomplete Elliptic Integrals of the first, second and third kinds. Besides the Legendre form given below, the elliptic integrals may also be expressed in Carlson symmetric form – which has been the basis for the implementation in the SandMath – completely based on the JMB_MATH ROM.

The incomplete elliptic integral of the first kind F is defined as:

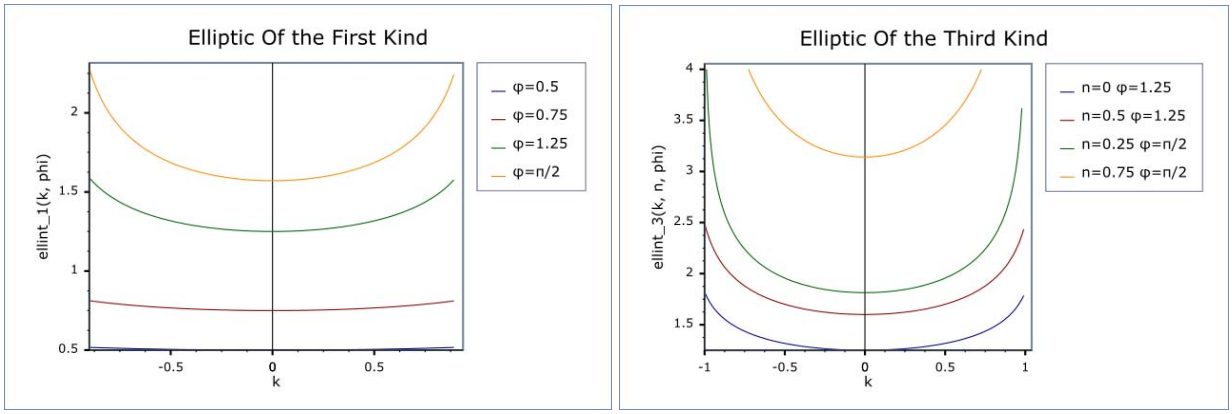
$$F(\varphi, k) = F(\varphi \,|\, k^2) = F(\sin\varphi; k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

which in terms of the Carlson Symmetric form $\mathbf{R_F}$, it results:

$$F(\phi, k) = \sin\phi R_F\left(\cos^2\phi, 1 - k^2\sin^2\phi, 1\right)$$

ELIPF is implemented as a MCODE function which simply calls **CRG** with the appropriate input parameters. All the heavy lifting is thus performed by **CRG**, which together with **CRJ** do all the hard work in the calculation for the Elliptic Integrals if first, second and third kinds.

The figure below shows the first and third kinds in comparison



This is a perhaps a good moment to define the Carlson symmetric forms. The Carlson symmetric forms of elliptic integrals are a small canonical set of elliptic integrals to which all others may be reduced. They are a modern alternative to the Legendre forms. The Legendre forms may be expressed in terms of the Carlson forms and vice versa.

The Carlson Symmetric Elliptic integrals of the First and Third kinds are defined as:

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}$$

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty \frac{dt}{(t+p)\sqrt{(t+x)(t+y)(t+z)}}$$

CRF and **CRJ** are the functions in the SandMath that calculate their values. The arguments are expected to be in the stack registers, and the result will be placed in x upon completion.

The term symmetric refers to the fact that in contrast to the Legendre forms, these functions are unchanged by the exchange of certain of their arguments. The value of $\mathbf{R_F}$ is the same for any permutation of its arguments, and the value of $\mathbf{R_J}$ is the same for any permutation of its first three arguments.

The Carlson Symmetric Elliptic integral of the 2nd. Kind is defined as:

$$R_G(x,y,z) = \frac{1}{4}\int_0^\infty \frac{t}{\sqrt{(t+x)(t+y)(t+z)}}\left(\frac{x}{t+x}+\frac{y}{t+y}+\frac{z}{t+z}\right)dt.$$

And is calculated using the following expression involving **CRF** and **CRJ**:

$$2.R_G(x;y;z) = z.R_F(x;y;z) - (x-z)(y-z)/3\ R_D(x;y;z) + (\ x.y/z\ )^{1/2}$$

Airy Functions.

For real values of x, the Airy function of the first kind is defined by the improper integral

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3 + xt\right)\,dt,$$

which converges because the positive and negative parts of the rapid oscillations tend to cancel one another out (as can be checked by integration by parts).

The Airy function of the second kind, denoted Bi(x), is defined as the solution with the same amplitude of oscillation as Ai(x) as x goes to −∞ which differs in phase by π / 2:

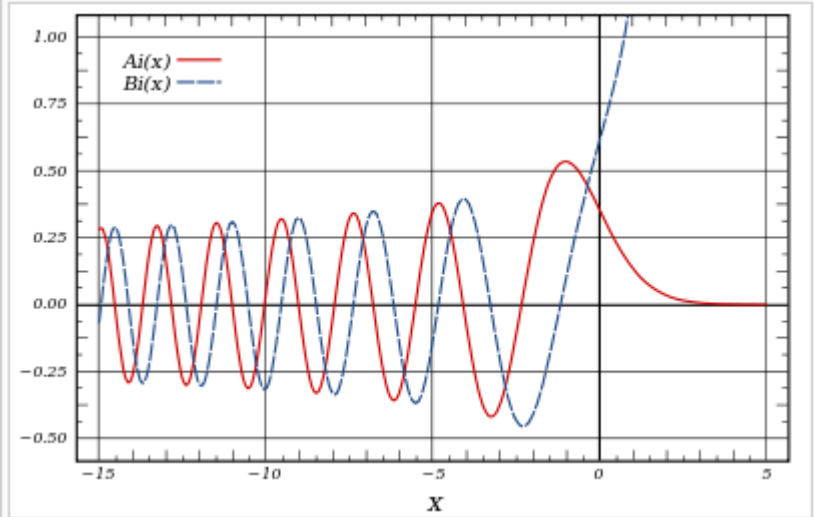
$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\tfrac{1}{3}t^3 + xt\right) + \sin\left(\tfrac{1}{3}t^3 + xt\right)\right]\,dt.$$

The expressions used to program them are again based on HGF+, as follows:

$$\mathrm{Ai}(x) = [3^{-2/3} / \Gamma(2/3)]\ {}_0F_1(\ ; 2/3; x^3/9) - x\,[3^{-1/3} / \Gamma(1/3)]\ {}_0F_1(\ ; 4/3; x^3/9)$$

$$\mathrm{Bi}(x) = [3^{-1/6} / \Gamma(2/3)]\ {}_0F_1(\ ; 2/3; x^3/9) + x\,[3^{1/6} / \Gamma(1/3)]\ {}_0F_1(\ ; 4/3; x^3/9)$$

The figure below shows Ai and Bi plotted for -15 < x < 5



REGISTERS: R00 thru R04
FLAGS: none

Stack	Input	Output
Y	n/a	Bi(x)
X	x	Ai(x)

Example:

0.4 **ΣFL$** "**AIRY**" -> Ai(0.4) = 0.254742355 ; or: [ΣFL], [**H**], [**Y**]
X<>Y -> Bi(0.4) = 0.801773001

Fresnel Integrals.

Fresnel integrals, S(x) and C(x), are two transcendental functions named after Augustin-Jean Fresnel that are used in optics. They arise in the description of near field Fresnel diffraction phenomena, and are defined through the following integral representations:

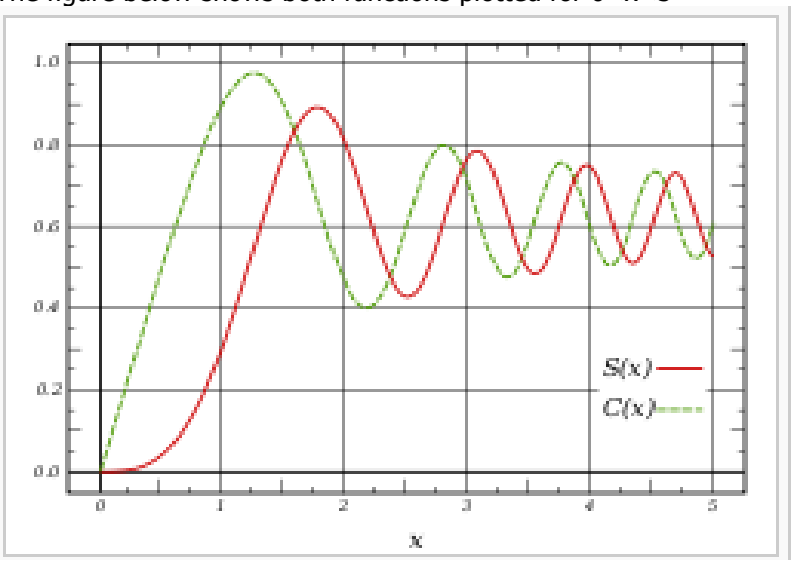
$$S(x) = \int_0^x \sin(t^2)\,dt, \quad C(x) = \int_0^x \cos(t^2)\,dt.$$

The function **CSX** will calculate both S(x) and C(x) for the argument in X, returning the results in Y and X respectively. It is a short FOCAL program that uses (yes you guessed it) the Generalized Hyper-geometric function, according to the expressions:

$$S(x) = (\pi\, x^3/6)\ {}_1F_2(3/4 ; 3/2 , 7/4 ; -\pi^2\, x^4/16),$$ and

$$C(x) = x\ {}_1F_2(1/4 ; 1/2 , 5/4 ; -\pi^2\, x^4/16)$$

The figure below shows both functions plotted for 0<x<5



REGISTERS: R00 thru R04
FLAGS: none

Stack	Input	Output
Y	n/a	S(x)
X	x	C(x)

Examples:

1.5 **ΣFL$** "**CSX**" -> C(1.5) = 0.445261176 X<>Y, S(1.5) = 0.697504960
4 **ΣFL$** "**CSX**" -> C(4) = 0.498426033 X<>Y, S(4) = 0.420515754

Or: [ΣFL], [**H**], [**C**]

Weber and Anger functions – WEBAN.

In mathematics, the Anger function, introduced by C. T. Anger (1855), is a function defined as

$$\mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\,d\theta$$

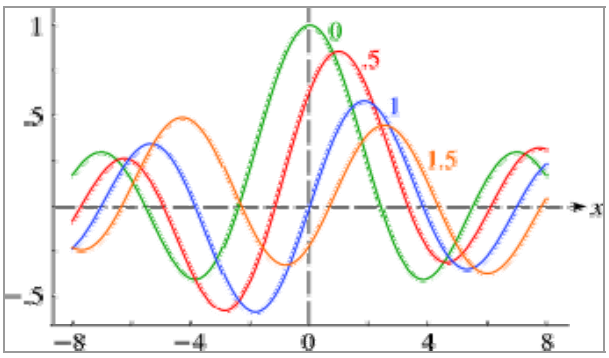
The Weber function introduced by H. F. Weber (1879), is a closely related function defined by:

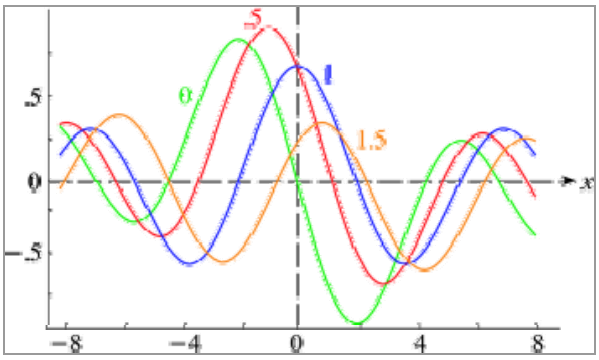
$$\mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(\nu\theta - z\sin\theta)\,d\theta$$

If v is an integer then Anger functions **J**v are the same as Bessel functions **J**v, and Weber functions can be expressed as finite linear combinations of Struve functions (Hn and Ln).

With n and x in the stack, **WEBAN** will return both **J**(n,x) and **E**(n,x) in the Y and X stack registers respectively.

The figures below show four of these functions for 4 orders(0, 0.5, 1, and 1,5) – Anger on the left plots, and Weber on the right. [Check: J(0,0) = 1, and E(0,0) = 1]





Note that **WEBAN** will return both values to the stack.

REGISTERS: R00 thru R06
FLAGS: none

Stack	Input	Output
Y	n	**J**(n,(x)
X	x	**E**(n,x)

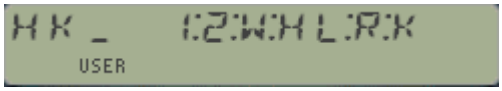
Example:

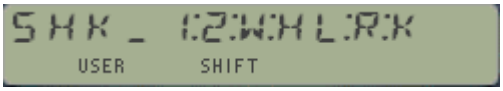
2 , SQRT, PI, **ΣFL$** "**WEBAN**" -> **E(**sqrt(2), π) = - 0.315594385
X<>Y -> **J(**sqrt(2), π) = 0.366086559

Or: [ΣFL], [**H**], [**W**]

3.5.2. Hankel, Struve, and similar functions.

The second sub-function launcher is the Hankel group. It's loosely centered on the Hankel functions, plus related sort. The launcher prompt is activated by pressing [**H**] at the main **ΣFL** prompt, and offers the following 14 choices – in two line-ups controlled by the [SHIFT] key. Note the different leadings on each screen, keeping the choices constant regardless:





The table below shows in the first column the letter used for each of the functions within this group:

[HK]	Function	Author	Description
[1]	**HK1**	*Ángel Martin*	Hankel1 Function
[2]	**HK2**	*Ángel Martin*	Hankel2 Function
[W]	**W0L**	*Ángel Martin*	Lambert W0
[H]	**HNX**	*JM Baillard*	Struve H Function
[L]	**LOML**	*JM Baillard*	Lommel s1 function
[R]	**LERCH**	*JM Baillard*	Lerch Transcendental function
[K]	**KLV**	*JM Baillard*	Kelvin Functions 1st kind
[1]	**SHK1**	*Ángel Martin*	Spherical Hankel1
[2]	**SHK2**	*Ángel Martin*	Spherical Hankel2
[W]	**W1L**	*Ángel Martin*	Lambert W1
[H]	**LNX**	*JM Baillard*	Struve Ln Function
[L]	**ALF**	*JM Baillard*	Associated Legendre function 1st kind - Pnm(x)
[R]	**TMNR**	*JM Baillard*	Toronto function
[K]	**KUMR**	*Ángel Martin*	Kummer Function

Here we finally find both branches of the Lambert W function, W0L and W1L, described previously in thie manual. Observant users would have noticed that the name in the previous section was WL0 instead – this is not a dyslexya-induced error, but an intended difference to tell both FAT entries apart. - That and also the limitation in the secondary FAT to have function names ending with numeric characters, but that's another story.

So your several choices in terms of launchers are as follows:-

a) Function **WL0** in main FAT

XEQ "**WL0**", the ordinary method
[**ΣFL**], [**M**], shortcut using the main launcher
ΣFL$** "**WL0**", since **ΣFL$ also finds functions in the main FAT
[**ΣFL**], [ALPHA], "**WL0**"

b) Functions **W0L** and **W1L** in secondary FAT

[**ΣFL**], [**H**], [**W**]
ΣFL# 032,
ΣFL$ "**W0L**"
[**ΣFL**], [ALPHA], "**W0**L"

[**ΣFL**], [**H**], [SHIFT], [**W**]
ΣFL# 033
ΣFL$ "**W1L**"
[**ΣFL**], [ALPHA], "**W1L**"

Now that's what I'd call both a digression and multiple ways to skin this cat.

Hankel functions – yet a Bessel third kind.

Another important formulation of the two linearly independent solutions to Bessel's equation are the Hankel functions Hα(1)(x) and Hα(2)(x), defined by:

$$H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(x)$$
$$H_\alpha^{(2)}(x) = J_\alpha(x) - iY_\alpha(x)$$

where i is the imaginary unit. These linear combinations are also known as Bessel functions of the third kind, and it's just an association of the previous two kinds together.

This definition allows for relatively simple programming only using the real-domain Bessel programs – assuming the individual results for J and Y are not complex. The small program in the next page shows the FOCAL code to just drive the execution of both **JBS** and **YBS**, piercing them together via **ZOUT** (or ZAWIEW in the 41Z module).

Getting Spherical, are we?

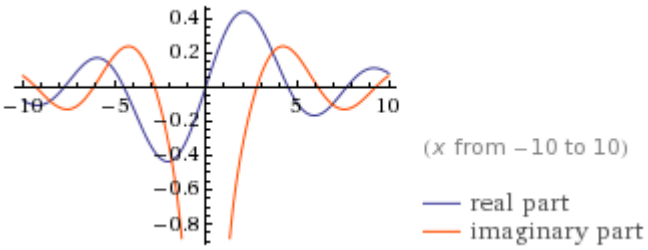
Finally, there are also spherical analogues of the Hankel functions, as follows:

$$h_n^{(1)}(x) = j_n(x) + iy_n(x)$$
$$h_n^{(2)}(x) = j_n(x) - iy_n(x).$$

The FOCAL programs below list the simple code snippets to program the three pairs of functions just covered, as follows:

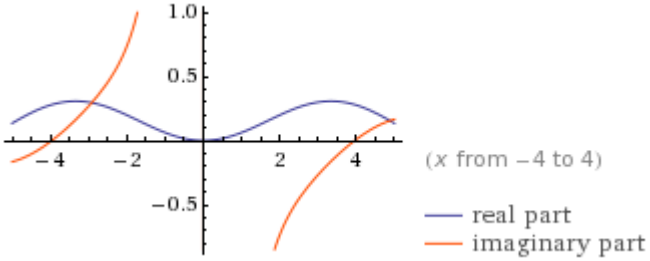
1. Hankel functions, HK1 and HK2
2. Spherical Bessel functions, SJBS and SYBS
3. Spherical Hankel functions, SHK1 and SHK2.

Note the symmetry in the code for the spherical programs, making good use of the stack efficiency derived from the utilization of the MCODE JBS function.



(*x* from −10 to 10)

real part
imaginary part



(*x* from −4 to 4)

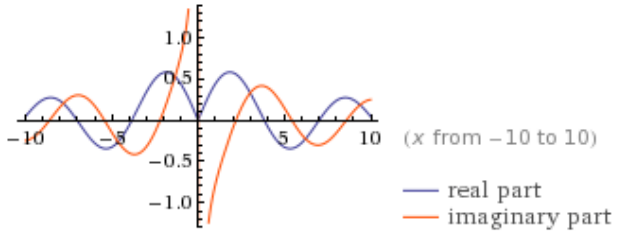
real part
imaginary part

The plots on the left show the Spherical Hankel-1 function for orders 1 and 2, for a short range of the argument x. Explicitly, the first few are

$$h_0^{(1)}(z) = -i\,e^{iz}\frac{1}{z}$$
$$h_1^{(1)}(z) = -e^{iz}\frac{z+i}{z^2}$$
$$h_2^{(1)}(z) = i\,e^{iz}\frac{z^2+3\,i\,z-3}{z^3}$$
$$h_3^{(1)}(z) = e^{iz}\frac{z^3+6\,i\,z^2-15\,z-15\,i}{z^4}.$$

1	LBL "HANK1"	
2	SF 03	
3	GTO 00	
4	LBL "HANK2"	
5	CF 03	
6	LBL 00	
7	JBS	
8	STO 04	
9	RCL Z	
10	RCL Z	n
11	ST+ X	x/2
12	YBS	
13	FC?C 03	
14	CHS	
15	RCL 04	
16	ZAVIEW	
17	END	

Plot of H(1)(1,x):



1	LBL "SHANK1"	
2	SF 04	
3	GTO 00	
4	LBL "SHANK2"	
5	CF 04	
6	LBL 00	
7	XEQ "SJBS"	
8	STO 04	
9	RCL 00	
10	RCL O	
11	ST+ X	
12	XEQ "SYBS"	
13	FC?C 04	
14	CHS	
15	RCL 04	
16	ZAVIEW	
17	END	

1	LBL "SJBS"	
2	SF 03	
3	GTO 00	
4	LBL "SYBS"	
5	CF 03	
6	LBL 00	
7	X<>Y	
8	STO 00	
9	0,5	
10	+	n+1/2
11	FC? 03	
12	CHS	-(n+1/2)
13	X<>Y	
14	JBS	
15	FC? 03	
16	GTO 00	
17	RCL 00	n
18	INCX	n+1
19	CHSYX	(-1)^(n+1) * JBS
20	LBL 00	
21	PI	
22	RCL O	x/2
23	ST+ X	
24	ST+ X	
25	/	
26	SQRT	
27	*	
28	END	

Examples.-

Calculate H1, H2, SH1, and SH2 for the following values in the table:

Arguments		H1	H2	SH1	SH2
n	x				
1	1	Z=0,440-J0,781	Z=0,440+J0,781	Z=0,301-J1,382	Z=0,301+J1,382
1	-1	*DATA ERROR*			
0.5	1	Z=0,671-J0,431	Z=0,671+J0,431	Z=0,552-J0,979	Z=0,552+J0,979
0.5	0.5	Z=0,541-J0,990	Z=0,541+J0,990	Z=0,429-J2,608	Z=0,429+J2,608
-0.5	1	Z=0,431+J0,671	Z=0,431-J0,671	Z=0,959+J0,111	Z=0,959-J0,111
-0.5	-1	*DATA ERROR*			
Shortcut:		[∑FL],[H], [1]	[∑FL],[H], [2]	[∑FL],[H],[SHIFT],[1]	[∑FL],[H],[SHIFT],[2]

Where we see that *for negative arguments* (integer and non-integer orders both), the result of the Bessel function of the second kind is itself a complex number, therefore the DATA ERROR message. Note also the symmetric nature of the values for each of the function pairs, H1 with H2, and SH1 with SH2.

Struve functions.

Struve functions are solutions y(x) of the non-homogenous Bessel's differential equation:

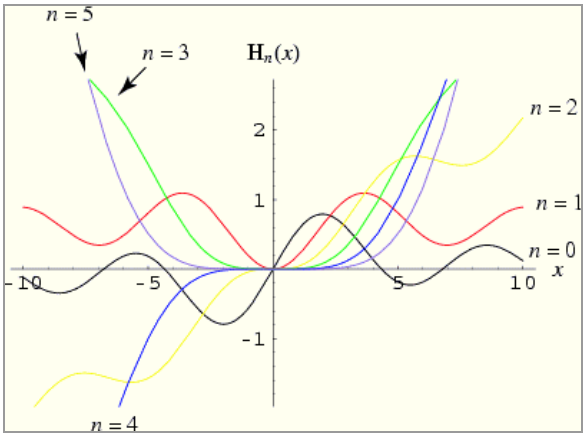
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=\frac{4(x/2)^{\alpha+1}}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$$

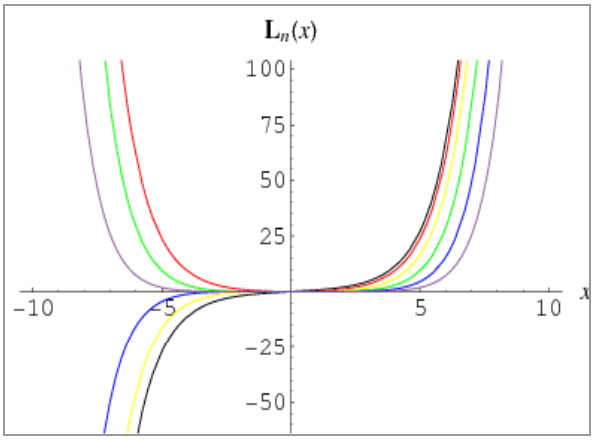
Struve functions H(n,x), and Modified Struve Functions L(n,x), have the following power series forms:

$$\mathbf{H}_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+\frac{3}{2})\Gamma(m+\alpha+\frac{3}{2})}\left(\frac{x}{2}\right)^{2m+\alpha+1}$$

$$\mathbf{L}_\nu(z)=\left(\frac{z}{2}\right)^{\nu+1}\sum_{k=0}^{\infty}\frac{1}{\Gamma(\frac{3}{2}+k)\Gamma(\frac{3}{2}+k+\nu)}\left(\frac{z}{2}\right)^{2k}$$

The figure below shows a few Struve functions or integer order, n=1 to 5; for –10<x<10





Struve functions of any order can be expressed in terms of the Generalized Hypergeometric function 1F2 (which is not the Gauss Hypergeometric function 2F1). – This is the expression used in the SandMath implementation:

$$\mathbf{H}_\alpha(z)=\frac{(z/2)^{\alpha+1/2}}{\sqrt{2\pi}\Gamma(\alpha+3/2)}{}_1F_2(1,3/2,\alpha+3/2,-z^2/4).$$

in other words, referred to the Rationalized Generalized Hypergeometric function (which with such a long name it definitely must be a formidable function… but it's just the same divided by Gamma)

$$H_n(x)=(x/2)^{n+1}\ {}_1F\tilde{}_2(\ 1\ ;\ 3/2\ ,\ n+3/2\ ;\ -x^2/4\ )$$

$$L_n(x)=(x/2)^{n+1}\ {}_1F\tilde{}_2(\ 1\ ;\ 3/2\ ,\ n+3/2\ ;\ x^2/4\ )$$

Examples: Compute **H**(1.2 , 3.4) and **L**(1.2 , 3.4)

1.2 ENTER^ , 3.4 **ΣFL$** "**HNX**" -> H(1.2 , 3.4) = 1.113372657
1.2 ENTER^ , 3.4 **ΣFL$** "**LNX**" -> L(1.2 , 3.4) = 4.649129471

Or: [ΣFL], [**H**], [**H**] for HNX, and: [ΣFL], [**H**], [SHIFT], [**H**] for LNX

Lommel functions.

The Lommel differential equation is an inhomogeneous form of the Bessel differential equation:

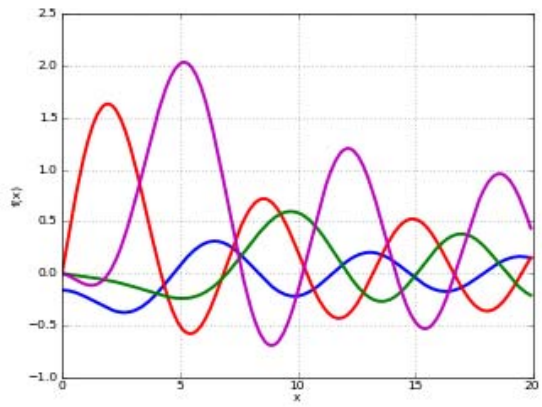
$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = z^{\mu+1}.$$

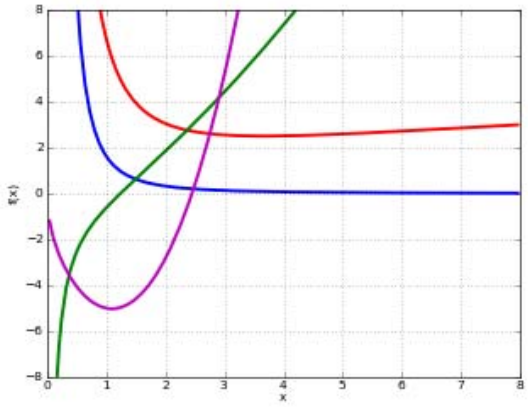
Two solutions are given by the Lommel functions sμ,ν(z) and Sμ,ν(z), introduced by Eugen von Lommel (1880),

$$s_{\mu,\nu}(z) = \frac{1}{2}\pi\left[Y_\nu(z)\int_0^z z^\mu J_\nu(z)\,dz - J_\nu(z)\int_0^z z^\mu Y_\nu(z)\,dz\right]$$

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) - \frac{2^{\mu-1}\Gamma(\frac{1+\mu+\nu}{2})}{\pi\Gamma(\frac{\nu-\mu}{2})}\left(J_\nu(z) - \cos(\pi(\mu-\nu)/2)Y_\nu(z)\right)$$

where Jv(z) is a Bessel function of the first kind, and Yv(z) a Bessel function of the second kind.





Using the Generalized Hypergeometric function the expressions for s1(m,n,x) is:

$$s^{(1)}_{m,n}(x) = x^{m+1} / [\,(m+1)^2 - n^2\,]\ {}_1F_2\left(1\ ;\ (m-n+3)/2\ ,\ (m+n+3)/2\ ;\ -x^2/4\right)$$

LOMMEL calculates s1 (m,n,x) = here are the specifics:

DATA REGISTERS: R00 thru R09: temp
Flags Used: F01

Stack	Input	Output
Z	m	/
Y	n	/
X	x	**s1(x)**

Example:

2 SQRT
3 SQRT
PI **ΣFL$** "**LOML**" -> s1[sqrt(2), sqrt(3), π) = 3.003060384

Or: [ΣFL], [**H**], [**L**] instead

Lerch (Transcendent) Function.

In mathematics, the Lerch zeta-function, sometimes called the Hurwitz–Lerch zeta-function, is a special function that generalizes the Hurwitz zeta-function and the polylogarithm. It is named after Mathias Lerch.

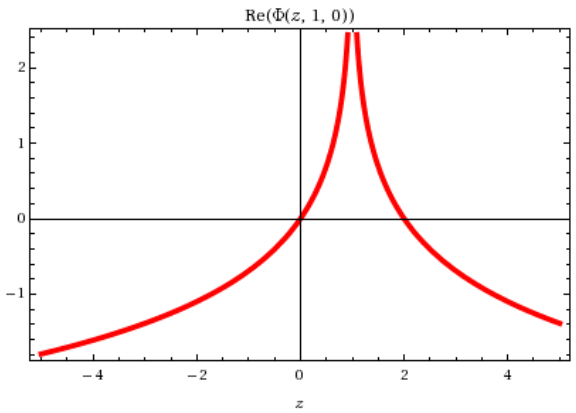
The Lerch zeta-function **L** and a related function, the Lerch Transcendent Φ, are given by:

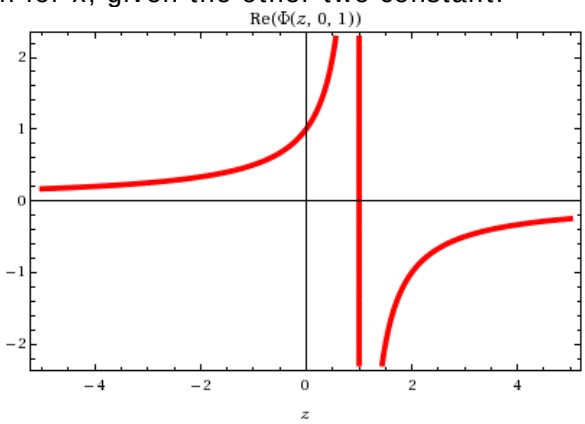
$$L(\lambda,\alpha,s)=\sum_{n=0}^{\infty}\frac{\exp(2\pi i\lambda n)}{(n+\alpha)^s}. \qquad \Phi(z,s,\alpha)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s}.$$

Special cases.- The Lerch Transcendent generates other special functions as particular cases, as it's shown in the table below:

The Hurwitz zeta-function	$\zeta(s,\alpha)=L(0,\alpha,s)=\Phi(1,s,\alpha).$
The Legendre chi function	$\chi_n(z)=2^{-n}z\Phi(z^2,n,1/2).$
The Riemann zeta-function	$\zeta(s)=\Phi(1,s,1).$
The polylogarithm	$\mathrm{Li}_s(z)=z\Phi(z,s,1).$
The Dirichlet eta-function	$\eta(s)=\Phi(-1,s,1).$

The figures below depict the representation for x, given the other two constant.





The SandMath implementation **LERCH** is for the Lerch Transcendent function. It is a short MCODE routine originally written by Jean-Marc Baillard, which calculates the series terms and adds them until they don't have a contribution to the final result. It is a slow converging series, and therefore the execution time can be rather long (at normal CPU speeds).

Data input follows the usual conventions for the stack registers, entering x as the last parameter (in register X) – despite the written form:

Stack	Input	Output
Z	s	T
Y	a	T
X	x	**Φ(x,s,a)**

Examples:-

PI ENTER^ , 0.6 ENTER^ , 0.7 **ΣFL$** "**LERCH**" - > Φ (0.7 ; π ; 0.6) = 5.170601130
3 ENTER^, -4.6 ENTER^ , 0.8 **ΣFL$** "**LERCH**" -> Φ (0.8 ; 3 ; -4.6) = 3.152827048

Or: [ΣFL], [**H**], [**R**] instead

Kelvin Functions. -

In applied mathematics, the Kelvin functions of the first kind -Berv(x) and Beiv(x) - and of the Second kind - Kerv(x) and Keiv(x) - are the real and imaginary parts, respectively, of

$J_\nu(xe^{3\pi i/4})$, for the 1st. Kind $K_\nu(xe^{\pi i/4})$ for the 2nd. Kind.

These functions are named after William Thomson, 1st Baron Kelvin

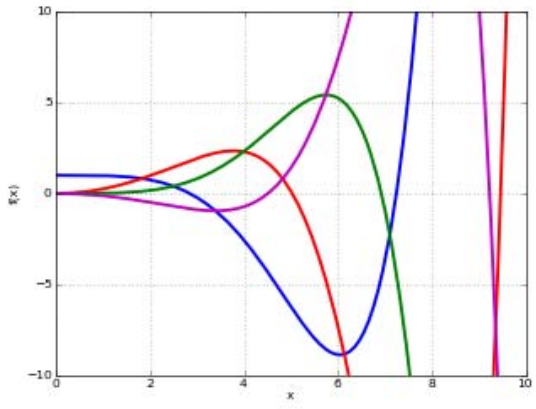
For integers n, Bern(x) and Bein(x) have the following series expansion

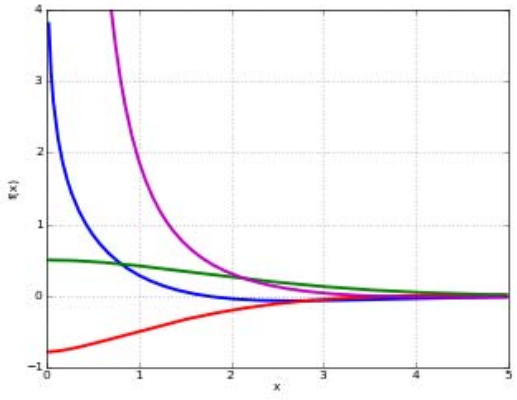
$$\mathrm{Ber}_n(x) = \left(\frac{x}{2}\right)^n \sum_{k\geq 0} \frac{\cos\left[\left(\frac{3n}{4}+\frac{k}{2}\right)\pi\right]}{k!\Gamma(n+k+1)}\left(\frac{x^2}{4}\right)^k$$

and

$$\mathrm{Bei}_n(x) = \left(\frac{x}{2}\right)^n \sum_{k\geq 0} \frac{\sin\left[\left(\frac{3n}{4}+\frac{k}{2}\right)\pi\right]}{k!\Gamma(n+k+1)}\left(\frac{x^2}{4}\right)^k$$

The figure below shows Ber(n,x) and Ker(n,x) for the first 4 integer orders and real arguments:





Ber(n,X) and Bei(n,x) are available in the SandMath, implemented as FOCAL programs written by JM Baillard. Both values are calculated simultaneously by **KLV**, and left in X,Y registers as follows:

Stack	Input	Output
Y	n	**bei(**n,x**)**
X	x	**ber(**n,x**)**

Example:

```
2   SQRT ,  PI   ΣFL$ "KLV"  ->   ber (sqrt(2), π) = -0.674095951
                      X<>Y   ->   bei (sqrt(2), π) = -1.597357210
```

Or: [ΣFL], [**H**], [**K**] instead

Kummer Function.

Kummer's equation has two linearly independent solutions M(a,b,z) and U(a,b,z).

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0.$$

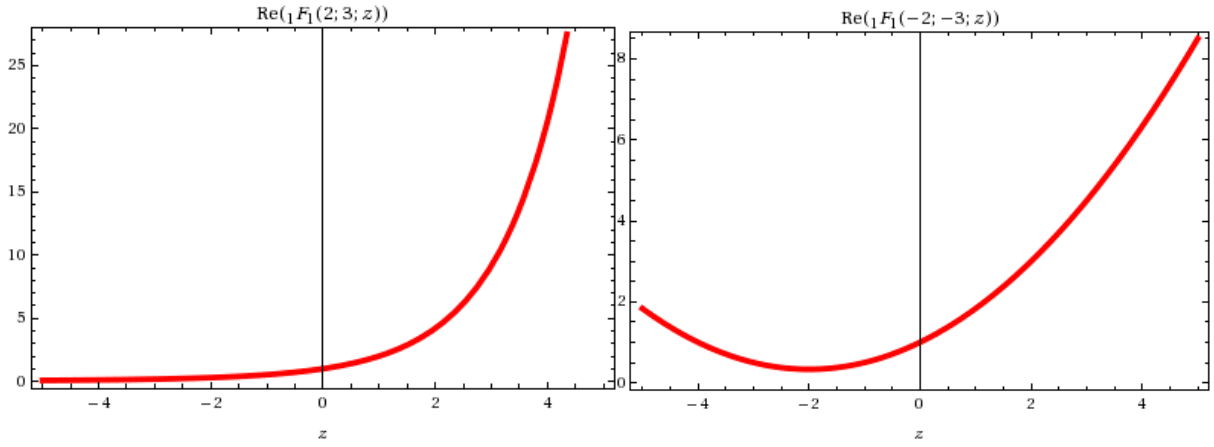
Kummer's function of the first kind M (also called Confluent Hypergeometric function) is a generalized hypergeometric series introduced in (Kummer 1837), given by

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} = {}_1F_1(a;b;z)$$

Where $a^{(n)}$ is the rising factorial, defined as:

$$a^{(n)} = a(a+1)(a+2)\cdots(a+n-1)$$

The figures below depict two particular cases for {a=2, b=3} and {a=-2, b=-3}



The SandMath implementation is got to be one of the simplest application ot **HGF+** possible, which renders acceptable accuracy to the results

DAT REGISTERS:

a – R00; b – R01

Stack	Input	Output
X	x	**M**(a;b;x)
L	/	x

Examples:

Compute M(2;3;-π) and M(2;3;-π)

2 STO 01 , 3 STO 02 , PI CHS, **ΣFL$** "**KUMR**" -> M(2;3;-π) = 0.166374562
2 STO 01 , 3 STO 02 , PI **ΣFL$** "**KUMR**" -> M(2;3;π) = 10,24518011

Or: [ΣFL], [**H**], [SHIFT], [**K**] instead

Associated Legendre Functions.

In mathematics, the Legendre functions P(λ), Q(λ) and associated Legendre functions Pμ(λ) and Qμ(λ) are generalizations of Legendre polynomials to non-integer degree. Associated Legendre functions are solutions of the Legendre equation:

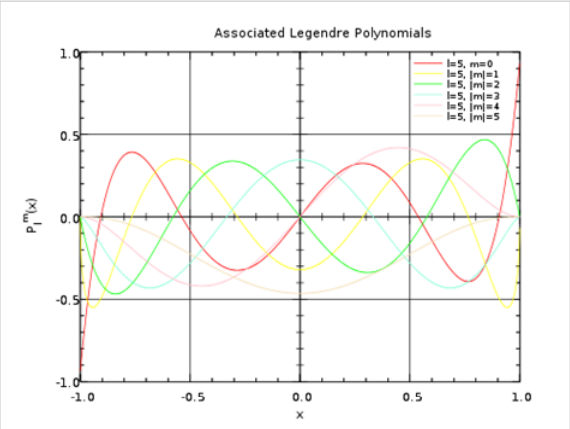
$$(1-x^2)\,y'' - 2xy' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-x^2}\right] y = 0,$$

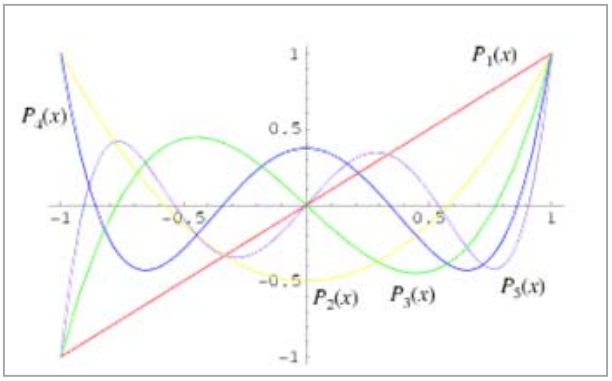
where the complex numbers λ and μ are called the degree and order of the associated Legendre functions respectively. *Legendre polynomials are the associated Legendre functions of order μ=0.*

These functions may actually be defined for general complex parameters and argument:

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+z}{1-z}\right]^{\mu/2} {}_2F_1(-\lambda,\lambda+1;1-\mu;\frac{1-z}{2}), \qquad \text{for } |1-z|<2$$

The figures below give a couple of graphical plots for the Legendre Polynomials:





REGISTERS: R00 thru R05
FLAGS: /

Stack	Input	Output
Z	m	/
Y	n	/
X	x	**P**(n,m.x)

Examples:

0.4 ENTER^, 1.3 ENTER^ , 0.7 **ΣFL$** "ALF" -> P1.3|0.4(0.7) = 0.274932821
-0.6 ENTER^ , 1.7 ENTER^ , 4.8 **ΣFL$** "ALF" - > P1.7|-0.6(4.8) = 10.67810281

Or: [ΣFL], [**H**], [SHIFT], [**L**] instead

Toronto Function.

In mathematics, the Toronto function T(m,n,r) is a modification of the confluent hypergeometric function defined by Heatley (1943) as

$$T(m,n,r) = r^{2n-m+1}e^{-r^2}\frac{\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+1)}{}_1F_1(\tfrac{1}{2}m+\tfrac{1}{2};n+1;r^2).$$

Which to untrained eyes just appears to be a twisted cocktail of the Kummer function, adding the exponential tot he mix and scaling it with Gamma.

DATA REGISTERS: R00 thru R04:
Flags: none.

Stack	Input	Output
Z	m	/
Y	n	/
X	r	T (m,n,r)

Example:

2 SQRT, 3 SQRT , PI, **ΣFL$** "**TMNR**" >>>> **T**(sqrt(2),sqrt(3), π) = 0.963524225

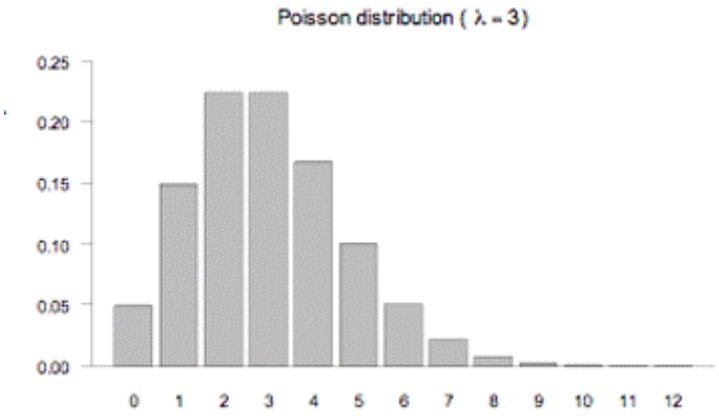
Or: [ΣFL], [**H**], [SHIFT], [**R**] instead

Poisson Standard Distribution. (*)

PSD is another Statistical function, which calculates the Poisson Standard Distribution. In probability theory and statistics, the Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time and/or space if these events occur with a known average rate and independently of the time since the last event

A discrete stochastic variable X is said to have a Poisson distribution with parameter λ>0, if for k = 0, 1, 2, ... the probability mass function of X is given by:

$$f(k;\lambda) = \Pr(X=k) = \frac{\lambda^k e^{-\lambda}}{k!},$$



Its inputs are k and λ in stack registers Y and X. **PSD**'s result is the probability corresponding to the inputs.

(*) Note that this function is not in the Hankel launcher, technically it belongs to the Statistics section.

3.4.3. Orphans and dispossessed.

The last group of sub-functions include those not belonging to any particular launcher – for no other particular reason that there's no more available space in the ROM. – Keep in mind that the only way to execute them is using the **ΣFL#** and **ΣFL$** launchers.

	Function	Author	Description
	-SP FNC	*Ángel Martin*	Cat header - does FCAT
	#BS	*Ángel Martin*	Aux routine, All Bessel
	#BS2	*Ángel Martin*	Aux routine - 2nd. kind, Integer orders
	AWL	*Ángel Martin*	Inverse Lambert
	DAW	*JM Baillard*	Dawson integral
	DBY	*JM Baillard*	Debye functions
	HGF	*JM Baillard*	Hyper-geometric function
	ITI	*Ángel Martin*	Integral if IBS
	ITJ	*Ángel Martin*	Integral of JBS
	LI	*Ángel Martin*	Logarithmic Integral
	PSD	*Ángel Martin*	Poisson Standard Distribution
	RHGF	*JM Baillard*	Regularized Hyper-geometric function
	SAE	*JM Baillard*	Surface Area of an Ellipsoid
	FCAT	*Ángel Martin*	Function Catalogue

Let's tackle the simpler ones on the list first.

FCAT (and **–SP FNC**) are usability enhancements for the admin and housekeeping. It invokes the sub-function CATALOG, with *hot-keys for individual function launch and general navigation*. Users of the POWERCL Module will already be familiar with its features, as it's exactly the same code – which in fact resides in the Library#4 and it's reused by both modules (so far).

#BS and **#BS2** are auxiliary functions used in the FOCAL programs for the Bessel functions of 2nd Kind, **KBS** and **YBS**. They were explained in more detail in the Bessel Functions paragraph.

AWL is the Inverse Lambert W function, an immediate application of the W definition involving just the exponential – but with additional accuracy when using the 13-digit routines in MCODE.
AWL = W * exp(W)

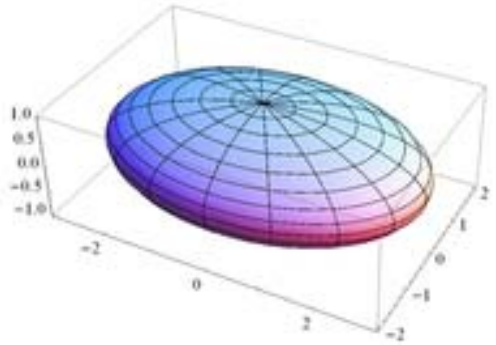
LI is the Logarithm Integral, also a quick application of the **EI** function, using the formula: Li(x) = Ei [(ln(x)] (see description for **EI** before). Note how **LI** starts as a MCODE functions that transfers into the FOCAL code calculating **EI,** so strictly speaking it's a sort of "hybrid" natured function.

SAE is a direct application of th Carlson Symmetric Integral of second kind, RG to calculate the surface aerea of an escalene ellipsoid (i.e. not of revolution):

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

which formula is:

Area = $4\pi.\mathbf{R_G}(a^2b^2 , a^2c^2 , b^2c^2)$



Let's now continue with the not-so-simple functions left, where some of them will – not surprisingly – be based on the Hyper-geometric functions again.

Debye Function.

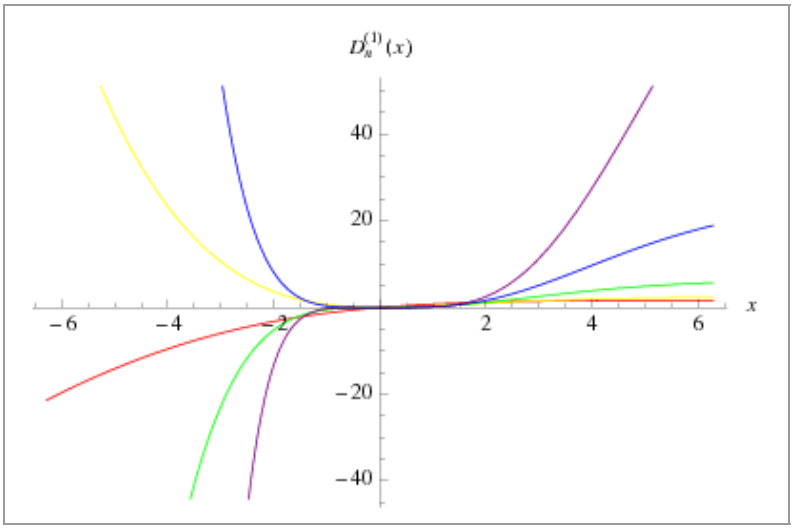
The family of Debye functions is defined by:

$$D_n(x) = \frac{n}{x^n} \int_0^x \frac{t^n}{e^t - 1}\, dt.$$

The functions are named in honor of Peter Debye, who came across this function (with n = 3) in 1912 when he analytically computed the heat capacity of what is now called the Debye model.

The formula used for n positive integers and X>0 is:

$$db(x;n) = \Sigma_{k>0}\ e^{-k.x}\ [\ x^n/k + n.x^{n-1}/k^2 + ... + n!/k^{n+1}\ ]$$



Despite being a FOCAL program, **DEBYE** pretty much behaves like an MCODE function: no data registers are used (only the stack and ALPHA), and the original argument preserved in LASTx. – credit is due to JM Baillard once more.

Stack	Input	Output
Y	n	**n**
X	x	**db**(n,x)
L	-	**x**

Example:

3 ENTER^, 0.7 , **ΣFL$** "**DEBYE**" -> DB(0.7 ; 3) = 6.406833597

Dawson Integral.

The Dawson function or Dawson integral (named for John M. Dawson) is either:

$$F(x) = D_+(x) = e^{-x^2} \int_0^x e^{t^2}\, dt$$

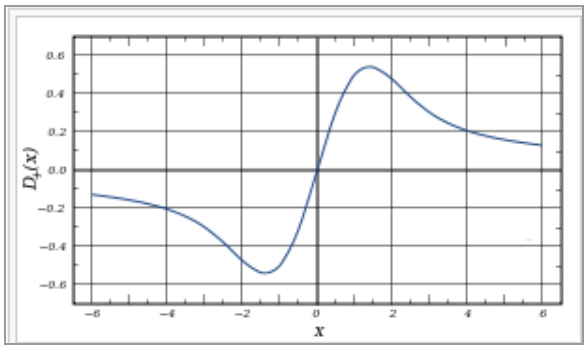
or:

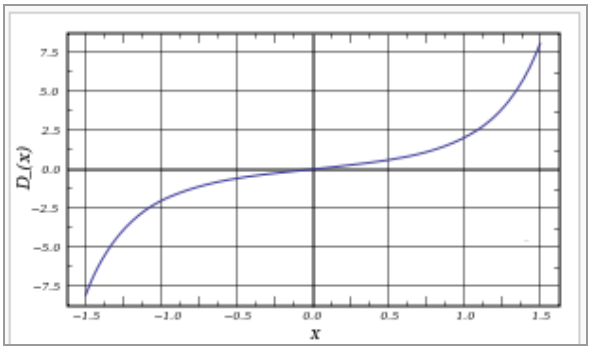
$$D_-(x) = e^{x^2} \int_0^x e^{-t^2}\, dt.$$

DAW computes F(x) by a series expansion:

$$F(x) = e^{-x^2} [\, x + x3/3 + x5/(5\ 2!) + x7/(7\ 3!) + \ldots\ldots \,]$$

The figures below show both functions in graphical form:





Here as well no data registers are used (!)

Stack	Input	Output
X	x	D+(x)
L	-	**e-x^2**

Examples:

```
1.94,  ΣFL$ "DAW"   ->     F(1.94 ) =  0.3140571659
  10,         R/S   ->     F(10)    =  0.05025384716
  15,         R/S   ->     F(15)    =  0.03340790676
```

For x > 15 , there will be an OUT OF RANGE condition.

Hyper-geometric Functions.

HGF and **RHGF** are the ordinary and the Regularized Hyler-geometric functions.

The Gaussian or ordinary hypergeometric function **$_2F_1$(**a,b;c; z) is a special function represented by the hypergeometric series, that includes many other special functions as specific or limiting case. It is defined for |z| < 1 by the power series:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

provided that c does not equal 0, −1, −2, ... Here (q)n is the Pochhammer symbol, which is defined by:

$$ (q)_n = \begin{cases} 1 & \text{if } n = 0 \\ q(q+1)\cdots(q+n-1) & \text{if } n > 0 \end{cases} $$

Many of the common mathematical functions can be expressed in terms of the hypergeometric function, or as limiting cases of it. Some typical examples are:

$$ \ln(1+z) = z\,{}_2F_1(1,1;2;-z) $$

$$ (1-z)^{-a} = {}_2F_1(a,1;1;z) $$

$$ \arcsin z = z\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};z^2\right) $$

The relation **$_2F_1$** (a,b,c,x) = (1 - x) -a **$_2F_1$** (a, c-b, c ; -x/(1-x)) is used if x < 0

The Regularized Hypergeometric function has a similar expression for each summing term, just divided by Gamma of the corresponding Pochhamer symbol plus the index n.

REGISTERS: R01 thru R03. They are to be initialised before executing **HGF** or **RGHF**.
R00 is not used.

R01 = a, R02 = b, R03 = c

Stack	Input	Output
X	X< 1	**$_2F_1$** (a,b,c,x)

***HGF** Examples:*

- 1.2 STO 01, 2.3 STO 02 , 3.7 STO 03

0.4 **ΣFL$** "**HGF**" -> 1.435242953
-3 **ΣFL$** "**HGF**" -> 0.309850661

***RHGF** Examples:*

- 2 STO 01, 3 STO 02, -7 STO 03

0.4 , **ΣFL$** "**RHGF**" - > 5353330.290
-3, **ΣFL$** "**RHGF**" -> 2128.650875

Integrals of Bessel Functions.

One of the usual approaches is to use the following recurrent relations for the calculation

$$\int_0^x J_\nu(t)\,dt = 2\sum_{k=0}^{\infty} J_{\nu+2k+1}(x)\,,$$

With Re(n)>0. More specifically, for positive integer orders n=1,2,.... we have

$$\int_0^x J_{2n}(t)\,dt = \int_0^x J_0(t)\,dt - 2\sum_{k=0}^{n-1} J_{2k+1}(x)\,,$$

and also

$$\int_0^x J_{2n+1}(t)\,dt = 1 - J_0(x) - 2\sum_{k=1}^{n} J_{2k}(x)\,,$$

There's however another approach based (yes, here as well!) on the Generalized Hypergeometric function **HGF+**. In fact the applicability of this method extends to the Integro-Differential forms of the Bessel functions, and so could be used to calculate second primitives or derivatives as well.

The expressions used in the SandMath for functions **ITJ** and **ITI** are as follows:

$$\mathbf{D^{\mu}\, I_n(x) = K\, x^{n-\mu}\, \Gamma(n+1)\; {}_2\tilde{F}_3[(n+1)/2, (n+2)/2\,;\, (n+1-\mu)/2, (n+2-\mu)/2, n+1\,;\, x^2/4\,]}$$

$$\mathbf{D^{\mu}\, J_n(x) = K\, x^{n-\mu}\, \Gamma(n+1)\; {}_2\tilde{F}_3[(n+1)/2, (n+2)/2\,;\, (n+1-\mu)/2, (n+2-\mu)/2, n+1\,;\, -x^2/4\,]}$$

Where $\mathbf{K = 2^{\mu-2n}\, sqrt(\pi);\ and\ \mu = -1}$ for the integral (primitive)

Just in case you don't believe it, take a look at this WolframAlpha's link:
http://www.wolframalpha.com/input/?i=integrate+%28besselI%28n%2Cx%29%29

$$\int J_n(x)\,dx = 2^{-n-1}\, x^{n+1}\, \Gamma\left(\frac{n+1}{2}\right) {}_1\tilde{F}_2\left(\frac{n+1}{2}\,;\, n+1,\, \frac{n+3}{2}\,;\, -\frac{x^2}{4}\right) + \text{constant}$$

$$\int I_n(x)\,dx = 2^{-n-1}\, x^{n+1}\, \Gamma\left(\frac{n+1}{2}\right) {}_1\tilde{F}_2\left(\frac{n+1}{2}\,;\, n+1,\, \frac{n+3}{2}\,;\, \frac{x^2}{4}\right) + \text{constant}$$

Nothing short of magical if you ask me – what I'd call "going out with a bang".

Examples:

1.4 ENTER^, 3, **ΣFL$** "**ITJ**" -> §|0,3 J(1.4,x).dx = 1.049262785

1.4 ENTER^, 3, **ΣFL$** "**ITI**" -> §|0,3 I(1.4,x).dx = 2.918753200

1 ENTER^, 3, **ΣFL$** "**ITJ**" -> §|0,3 J(1,x).dx = 1.260051955

0 ENTER^, 10, **ΣFL$** "**ITJ**" -> §|0,10 J(0,x).dx = 1.067011304

50 ENTER^, 30, **ΣFL$** "**ITJ**" -> §|0,30 J(50,x).dx =1.478729947 E-8

Appendix 11- Looking for Zeros.

Once again just connecting the dots: a brute-force crude implementation of a root finder for Bessel functions, made possible once the major task (i.e. calculating the function value) is reduced to a single MCODE function.

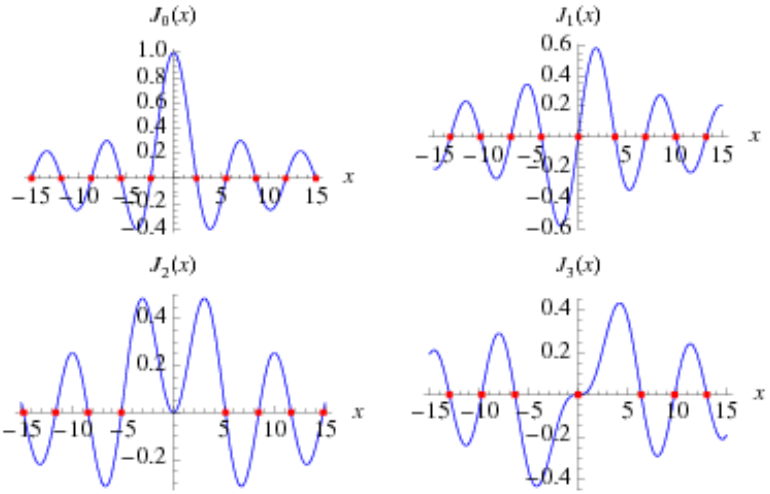
The following trivial-looking program (it really can't get any simpler!) uses **SOLVE** within the Advantage Pack, no less. Starting with zero, obvious guess values are the previous root and the root incremented by one. Successive repetitions will unearth all those roots; just make sure you have the "turbo" mode enabled on V41 (or equivalent emulator). Enjoy!

The first few roots **j**(n,k) of the Bessel function **J**n(x) are given in the following table for small nonnegative integer values of n and k

See also: *http://cose.math.bas.bg/webMathematica/webComputing/BesselZeros.jsp*

1	LBL "ITJBS"		
2	X<>Y		
3	STO 00		
4	X<>Y		
5	0		
6	X<>Y		
7	"JB"		
8	INTEG		
9	RTN		
10	LBL "JZER"		
11	-HL MATH	"Order=?"	
12	STOP		
13	STO 00		
14	0		
15	LBL 00		
16	RPLX		
17	3		
18	+		
19	"JB"		
20	SOLVE		
21	STOP		
22	INCX		
23	GTO 00		
24	LBL "JB"		
25	RCL 00		
26	X<>Y		
27	JBS		
28	END		

k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178



Note that the program listing also includes code to calculate the Integral of **JBS**, defined as incomplete function with the argument in the upper integration limit. Granted it isn't the fastest one in town but such isn't an issue on a modern-day emulator, and the economy of code cannot be stronger!

$$\int_0^x J_0(t)\,dt \qquad \int_0^x J_1(t)\,dt = 1 - J_0(x) \qquad \int_0^x J_n(t)\,dt$$

Which allegedly satisfies the equation: $\int_0^x \mathbf{J_n(t).dt} = 2\ (\ \mathbf{J_{n+1}(x)} + \mathbf{J_{n+3}(x)} + \mathbf{J_{n+5}(x)} + \ldots\ldots\ )$

This completes this manual.

Don't forget to check Jean-Marc Baillard extensive and authoritative references on the web (despite its unassuming web site name), located at: http://hp41programs.yolasite.com/

A treasure chest awaits you... enjoy the ride!